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A FOUR-DIMENSIONAL ANALOGUE OF PASCAL'S THEOREM FOR CONICS

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1. Introduction. As an extension of Pascal's theorem for conics, the following general problem may be investigated.

Let us consider in $[r]$* *all sets S of linear spaces given in number and dimensions, subject possibly to given incidence conditions; and a continuous system of algebraic varieties V. Supposing that no V contains the general S, one has to decide whether and how it is possible to express the condition for a set S to lie on a variety V, in the form of linear conditions for certain spaces, linearly deduced from the spaces of S taken in an arbitrary order.*

A less circumstantial problem for the particular case of ten points of an ordinary quadric was proposed in the year 1825 by the Academy of Brussels, and studied since then by several authors.† But the results obtained are intricate, and far from possessing the elegance of Pascal's theorem.

The case of seven points of a twisted cubic curve has been considered by Chasles, whose result is simple but unsymmetrical.‡ In addition one knows a few cases of a different, rather trivial kind. For instance, in ordinary space six skew lines lie on a cubic surface, if and only if they have a complementary sextuplet, forming a double-six with them.§

In the present paper I deal with the above problem in another particular case, giving for this a non-trivial extension of Pascal's theorem, in which all the aesthetic qualities are preserved. The result concerns five lines lying on a three-dimensional quadric of [4] (Part II). It suggests some interesting questions (§11), and easily yields applications of various types, both to plane and three-dimensional geometry (Part III). Some preliminary questions on double-fours of lines, having some interest also in themselves and leading to another extension of Pascal's theorem, are studied in Part I.

Part I. On Double-Fours of Lines

2. The three types of double-fours of lines. We say that two quadruplets of lines a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 constitute a *double-four*, if both the lines a and the lines b are two by two skew, while a line a and a line b are skew or incident according as they have or have not the same index. Then *e.g.* the lines a_1, a_2, a_3 are incident with b_4, so that the linear space joining these four lines can only have dimension 3 or 4. From the incidences among the lines a and b, we see that this linear space contains also b_1, b_2, b_3 and a_4. Hence:

* The symbol $[r]$ designates a projective space of r dimensions. An $[r-1]$ in an r-dimensional space is hereinafter called a "prime" of this space.

† *Cf.* the interesting paper by H. W. Turnbull and A. Young in Trans. Cambridge Phil. Soc. 23, 1923–28, pp. 265–301, where extensive historical references are also given.

‡ M. Chasles, Aperçu historique, Paris 1865 (reprinted from the Mém. cour. par l'Ac. de Bruxelles, 1837), p. 404.

§ *Cf. e.g.* B. Segre, The non-singular cubic surface, Oxford, 1942, pp. 7, 35.

A double-four of lines lies either in a four-dimensional space or in an ordinary space.

In the first case the double-four will be called of *the first kind.* In the second case the double-four will be called of *the second* or *third kind*, according as it lies, or does not lie on a cubic surface.

3. The double-fours of the first kind. We obtain the more general double-four of the first kind, by taking four lines a_1, a_2, a_3, a_4 in a general position in [4], and considering the lines b_1, b_2, b_3, b_4 uniquely defined by the conditions of meeting $a_2a_3a_4$, $a_3a_4a_1$, $a_4a_1a_2$, $a_1a_2a_3$ respectively. For *e.g.* the lines a_2, a_3, a_4 are two by two skew and do not lie in a prime, so that they have a single common transversal b_1, the intersection of the primes joining them in pairs. Moreover, it is easily seen that the lines b_1, b_2, b_3, b_4 are two by two skew, and are also skew to a_1, a_2, a_3, a_4 respectively. The two quadruplets $a_1a_2a_3a_4$, $b_1b_2b_3b_4$ clearly form a double-four of the first kind; their relation is a reciprocal one, and they will be said to be mutually *complementary.*

Since the lines in [4] constitute an irreducible ∞^6 system, and a double-four is uniquely defined by either of its two complementary quadruplets, it follows that:

The double-fours of the first kind, of a given four-dimensional space, form an irreducible system of dimension 24.

This agrees with the remark that *any two such double-fours are transformed one into the other by* $2\cdot 4! = 48$ *collineations.* The double-four $(a_1a_2a_3a_4, b_1b_2b_3b_4)$ is transformed into another given one $(a_1'a_2'a_3'a_4', b_1'b_2'b_3'b_4')$, *e.g.* by the collineation defined by the conditions of transforming the six points a_2b_3, a_3b_2, a_3b_1, a_1b_3, a_1b_2, a_2b_1, into the six points $a_2'b_3'$, $a_3'b_2'$, $a_3'b_1'$, $a_1'b_3'$, $a_1'b_2'$, $a_2'b_1'$ respectively.

We give now a new and simple proof of the following well known results, due to C. Segre.*

If $d = (a_1a_2a_3a_4, b_1b_2b_3b_4)$ *is a double-four of the first kind, the four primes* a_1b_1, a_2b_2, a_3b_3, a_4b_4 *have a line in common. This line is therefore uniquely defined by either of the complementary quadruplets of d, and is said to be* associated *to it. The planes meeting four general lines in* [4] *constitute an irreducible* ∞^2 *system, and each of them also meets the associated line.*

It is easily seen that, since d is of the first kind, the primes a_1b_1, a_2b_2, a_3b_3 cannot have a plane in common, so that they intersect in a line r. The results stated will follow, if we prove that the general point O of r lies in the prime a_4b_4; and that there are ∞^1 planes through O meeting $a_1a_2a_3a_4$, and ∞^1 planes through O meeting $b_1b_2b_3b_4$.

Now all this is immediately seen, on projecting the lines a_i, b_i from O upon a prime, into a_i', b_i' say $(i = 1, 2, 3, 4)$. For the lines a_1', a_2', a_3' are two by two skew, and each of them meets b_1', b_2', b_3'. Therefore these six lines lie on an ordi-

* *Cf.* C. Segre, Alcune considerazioni elementari sull'incidenza di rette e piani nello spazio a quattro dimensioni, Rend. Circ. Mat. Palermo 2, 1888, p. 45, or H. F. Baker, Principles of geometry, IV, Cambridge, 1925, pp. 113–123.

nary quadric Q. This contains a_4', b_4' as generators of opposite systems, since a_4' meets b_1', b_2', b_3', and b_4' meets a_1', a_2', a_3'. Hence a_4' and b_4' are incident, and so the prime a_4b_4 contains in fact the point O. The planes projecting from O the two systems of generators of Q, give the two ∞^1 systems of planes having the required properties.

4. The general projection of a double-four of the first kind upon a prime. With the eight lines of a double-four of the first kind, $d=(a_1a_2a_3a_4, b_1b_2b_3b_4)$, one can form two mutually residual skew quadrilaterals, in the following three different ways: $a_1b_3a_2b_4$ and $b_1a_3b_2a_4$, $a_1b_4a_3b_2$ and $b_1a_4b_3a_2$, $a_1b_2a_4b_3$ and $b_1a_2b_4a_3$. Since each quadrilateral lies in a prime, we see that d lies in three quadrics, each consisting of two such primes. The eight lines of d obviously constitute the complete intersection of these three quadrics. Hence the latter determine a *net*, N say, formed by all the quadrics of [4] containing d.*

The general quadric of N is irreducible, and so contains ∞^3 lines. We thus obtain in [4] an ∞^5 *algebraic complex of lines*, K say, formed by the lines lying on some quadric of N. We shall prove that:

The complex K is of the third order, and contains all the lines meeting one or other of the lines of the double-four d.

The second part of this theorem is obvious, since a quadric of N containing two points generally chosen upon a line meeting the base curve d of N, clearly contains this line. We prove the first part, by noticing that the lines of K lying on a general plane π constitute an envelope of the third class, which is in fact the Cayleyan envelope of the net of conics cut out by N on π.

Moreover, from a well known property of the Cayleyan envelope, it follows that the lines of K are those lines of [4] on which N cuts out pairs of an ordinary involution. Hence:

The lines of K can be characterized by the property that the pairs of primes a_1a_2 and a_3a_4, a_1a_3 and a_4a_2, a_1a_4 and a_2a_3 cut out on them three pairs of points in involution.

From a previous result we see that the lines of K passing through a given general point O constitute a *cubic three-dimensional cone, containing the planes projecting the eight lines of d from O*. Hence, on account of §2, we can say that:

The general projection of a double-four of the first kind upon a prime is a double-four of the second kind.

We shall reciprocate this result in §7.

5. The prime sections of a double-four of the first kind. Let us consider in [4] a general prime, ω say, and denote by A_i, B_i its intersections with a_i, b_i respectively ($i=1, 2, 3, 4$). The lines $a_ia_jb_hb_k$, where i, j, h, k is any arrangement of the numbers 1, 2, 3, 4, are in a prime (§4); this intersects ω in a plane, on which the points $A_iA_jB_hB_k$ must consequently lie. Hence the lines A_iA_j and B_hB_k are incident.

* Such a net N can clearly be obtained, in two different ways, as the totality of the quadrics of [4] containing four general lines.

On calling *incident* two tetrahedra $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ when each edge A_iA_j of the former is incident with the complementary edge B_hB_k of the latter, we now prove the following theorem:

Two incident tetrahedra $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ lie in the same ordinary space. Moreover, their eight vertices are associated with respect to the quadrics of this space (i.e., are the base points of a net of quadrics).

The ordinary spaces α, β of the two tetrahedra have in common the points of incidence of the six pairs of complementary edges A_iA_j, B_hB_k. Hence α and β coincide, since these six points are neither collinear nor coplanar. For, on the one hand, there is no line meeting all the edges of a tetrahedron. On the other hand, if the six points A_iA_j, B_hB_k lie on a plane π, which is therefore contained both in α and in β, then π cannot be a common face of the two tetrahedra; for, otherwise, the edges of these tetrahedra lying on π would give two triangles, each inscribed in the other, and two such triangles do not exist. Hence π is distinct from the faces of at least one tetrahedron, say of $A_1A_2A_3A_4$. It follows that π intersects the plane $A_2A_3A_4$ along a line, which contains the points where A_3A_4, A_4A_2, A_2A_3 meet B_1B_2, B_1B_3, B_1B_4 respectively. Since the last three edges are non-coplanar, we see that B_1 must lie on π. The same conclusion can likewise be reached with regard to B_2, B_3, B_4, so that the four vertices of the tetrahedron $B_1B_2B_3B_4$ lie on π. This contradiction proves the first part of the the theorem.

The second part follows at once, on noticing that the eight vertices of the two tetrahedra can be distributed in three different ways in two quadruplets of coplanar points:

$A_1A_2B_3B_4$ and $B_1B_2A_3A_4$, $A_1A_3B_4B_2$ and $B_1B_3A_4A_2$, $A_1A_4B_2B_3$ and $B_1B_4A_2A_3$.

Hence the eight points lie in three reducible quadrics, not belonging to a pencil, and are the base points of the net determined by them.

From the previous results it follows that:

The two complementary quadruplets of a double-four of the first kind cut out the vertices of two incident tetrahedra on the general prime of their space. This is the only means of obtaining two incident tetrahedra, by taking as vertices eight points lying on the eight lines of the double-four.

We shall say that such a pair of incident tetrahedra is *inscribed in the double-four.*

6. The double-fours of the second and third kinds. We determine the more general double-four of an ordinary space ω, by taking four generic lines $a_1a_2a_3a_4$ in ω, and further choosing the lines b_1, b_2, b_3, b_4 as generators of the quadrics defined by the directrices $a_2a_3a_4$, $a_3a_4a_1$, $a_4a_1a_2$, $a_1a_2a_3$ respectively. The number of parameters on which the double-four depends is therefore

$$4 + 4 + 4 + 4 + 1 + 1 + 1 + 1 = 20.$$

The cubic surfaces in ω constitute an ∞^{19} linear system. Moreover, the general cubic surface contains 27 lines, and a finite number ($=540$) of double-fours,

which are all equivalent with respect to the group of the 27 lines.† Hence:

The double-fours of the second and third kinds of an ordinary space constitute two irreducible systems, of dimensions 19 *and* 20 *respectively; the latter system contains the former one.*

There is consequently a *single condition*, expressing that a double-four $d=(a_1a_2a_3a_4,\ b_1b_2b_3b_4)$ in ordinary space is of the second kind, *i.e.* lies on a cubic surface. The condition can be stated as an *incidence condition* as follows. There are two lines, say b', b'', meeting $a_1a_2a_3a_4$, and two lines, say a', a'', meeting $b_1b_2b_3b_4$. A cubic surface F containing d must obviously also contain the lines a', a'', b', b'', and the two sextuplets of lines a and b constitute a double-six on it.† Hence *the line a' meets either b' or b'', and the line a'' then meets b'' or b' respectively.* Conversely, it is easily seen that if one of these two incidence conditions holds, then d lies on a cubic surface, and the other condition also holds.

It is worth noticing that the lines a', a'', b', b'' are not linearily deducible from the double-four; hence the above criterion for d to lie on a cubic surface does not give a Pascal's theorem, in the sense of the Introduction. We shall obtain a theorem having the requisite property, at the end of next paragraph. Now we remark that:

A double-four of lines in ordinary space cannot lie on two different cubic surfaces.

For, otherwise, the two cubic surfaces would have 12 lines in common, and so they would have a common component. Now this is impossible, since it is easily seen that no double-four of lines lies on any reducible cubic surface.

7. A Pascal's theorem for double-fours in ordinary space. Let us consider in [4] a double-four of the first kind, $d=(a_1a_2a_3a_4,\ b_1b_2b_3b_4)$, a general point O and a general prime ω. We denote by α_i, β_i the planes projecting a_i, b_i from O, and by A_i, B_i, a_i', b_i' the intersections of ω with a_i, b_i, α_i, β_i respectively ($i=1, 2, 3, 4$). Then $d'=(a_1'a_2'a_3'a_4',\ b_1'b_2'b_3'b_4')$ is a double-four of the second kind lying in ω (§4), and $T=(A_1A_2A_3A_4,\ B_1B_2B_3B_4)$ is a pair of incident tetrahedra inscribed both in d and in d' (§5). We now prove the following theorem:

The double-four d' is the projection from O upon ω of an infinity of double-fours of the first kind lying in [4]; *these double-fours constitute an ∞^5 irreducible system, and can all be obtained by transforming d with the homologies of* [4] *of center O. There is an infinity of pairs of incident tetrahedra inscribed in d'; they constitute an ∞^4 irreducible system, and can all be obtained as projections from O upon ω of the prime sections of d.*

It is clear from §3 that the transforms of d by means of the ∞^5 homologies of [4] of center O constitute an ∞^5 irreducible system of double-fours of the first kind, each of which has d' as projection from O upon ω. Conversely, in order to determine in [4] a double-four $d^*=(a_1^*a_2^*a_3^*a_4^*,\ b_1^*b_2^*b_3^*b_4^*)$ of the first kind, having d' as projection from O upon ω, we can first of all choose b_4^* in a general posi-

† *Cf. e.g.* B. Segre, *op. cit.*, p. 7 and Ch. II.

tion on β_4. Then the lines a_1^*, a_2^*, a_3^* must lie in the planes α_1, α_2, α_3, and belong to the pencils having the points where b_4^* meets α_1, α_2, α_3 respectively as centers. We can choose a_1^*, a_2^*, a_3^* generally in such pencils, and then we have at most a single possibility for the remaining lines b_1^*, b_2^*, b_3^*, a_4^* of d^*, since these must contain the sets of points

$$(a_2^*\beta_1,\ a_3^*\beta_1), \qquad (a_1^*\beta_2,\ a_3^*\beta_2), \qquad (a_1^*\beta_3,\ a_2^*\beta_3), \qquad (b_1^*\alpha_4,\ b_2^*\alpha_4,\ b_3^*\alpha_4)$$

respectively. It follows that d^* can be uniquely defined as transform of d by means of the homology of center O, having b_4 and b_4^* as corresponding lines, and whose fundamental space is the join of the points $a_1a_1^*$, $a_2a_2^*$, $a_3a_3^*$, $b_4b_4^*$.

From §5, a general prime section of d gives a pair of incident tetrahedra inscribed in d; hence the projection of such a pair from O upon ω is a pair of incident tetrahedra inscribed in d'. Conversely, let $T^* = (A_1^*A_2^*A_3^*A_4^*, B_1^*B_2^*B_3^*B_4^*)$ be a pair of incident tetrahedra inscribed in d'. We shall shortly see that T^* is the section of ω with a double-four of the first kind, say d^*, having d' as projection from O upon ω. Hence d^* is transformed into d by a homology of center O. This homology transforms ω into a prime ω^*, and it is clear that T^* is the projection from O upon ω of the section of d with ω^*.

The result employed during the previous deduction is included in the following:

Let $d' = (a_1'a_2'a_3'a_4',\ b_1'b_2'b_3'b_4')$ be any double-four lying in an ordinary space ω, and $T = (A_1A_2A_3A_4,\ B_1B_2B_3B_4)$ be a pair of incident tetrahedra inscribed in d'. Then, on considering a space [4] *containing ω, and a point O lying in* [4] *but not in ω, it is possible to construct in* [4] *a double-four of the first kind having d' as projection from O upon ω, and intersecting ω in T.*

We denote by α_i, β_i the planes projecting a_i', b_i' from O $(i=1, 2, 3, 4)$, and by $d = (a_1a_2a_3a_4,\ b_1b_2b_3b_4)$ the required double-four. We can choose b_4 generally in β_4 through the point B_4. Then we define the lines a_1, a_2, a_3 as joins of the points (A_1, α_1b_4), (A_2, α_2b_4), (A_3, α_3b_4) respectively, the lines b_1, b_2, b_3 as joins of the points $(a_2\beta_1,\ a_3\beta_1)$, $(a_1\beta_2,\ a_3\beta_2)$, $(a_1\beta_3,\ a_2\beta_3)$ respectively, and the line a_4 as the line of [4] incident to b_1, b_2, b_3. Hence $d = (a_1a_2a_3a_4,\ b_1b_2b_3b_4)$ is a double-four of the first kind, having $d^* = (a_1'a_2'a_3'a_4^*,\ b_1'b_2'b_3'b_4')$ as projection from O upon ω, and $T^* = (A_1A_2A_3A_4^*,\ B_1^*B_2^*B_3^*B_4)$ as section with ω, where a_4^* is a line of ω, and B_1^*, B_2^*, B_3^*, A_4^* are points of ω, the three first of which lie on b_1', b_2', b_3' respectively. We need only to prove that B_1^*, B_2^*, B_3^*, A_4^*, a_4^* coincide with B_1, B_2, B_3, A_4, a_4' respectively.

From the incidence conditions for T and for T^*, we see that both B_1B_4 and $B_1^*B_4$ are lines through B_4 meeting A_2A_3; moreover, both these lines meet b_1', at B_1, B_1^* respectively. Since A_2A_3, b_1' are two skew lines, and B_4 lies on neither of them, it follows that $B_1 = B_1^*$. Likewise we see that $B_2 = B_2^*$, $B_3 = B_3^*$. Hence seven of the eight vertices of the pair of incident tetrahedra T are also vertices of T^*, so that (from §5) also the remaining vertices of T, T^* coincide, *i.e.* $A_4 = A_4^*$. Finally, from the incidence conditions for d', d^* it follows that a_4' and a_4^* coincide, since both these lines contain the point $A_4 = A_4^*$ and are incident with b_1', b_2', b_3'.

Let us consider again in [4] a prime ω and a point O not lying on ω. We know that the double-fours of the first kind of [4] form an ∞^{24} irreducible system (§3). In general, the projection of one of these double-fours from O upon ω is a double-four of the second kind (§4), and can be obtained by such a projection from exactly ∞^5 double-fours of the first kind lying in [4]. Hence the system formed by these projections has the dimension $24-5=19$, and therefore coincides with the ∞^{19} irreducible system formed by *all* the double-fours of the second kind lying in ω (§6). This proves the *converse* of the final result of §4. We see, moreover, that:

A double-four of lines lying in ordinary space is of the second kind (i.e. it lies on a cubic surface) if, and only if, there is a pair of incident tetrahedra inscribed in it. There are then ∞^4 such pairs of tetrahedra, and we determine uniquely one of them, by choosing four of their eight vertices in general positions on any four lines of the double-four.

This result gives a *linear test* for ascertaining whether a double-four of lines in ordinary space lies on a cubic surface. Hence, according to the Introduction, it can be considered as a *generalized Pascal's theorem.*

PART II. QUINTUPLETS OF RELATED LINES IN [4]

8. Pascal's theorem for five related lines in [4]. There are ∞^{14} three-dimensional quadrics in [4], and three conditions are required for a quadric to contain a given line. Hence five lines in [4] do not generally lie on a quadric, and *a single condition* is required for the existence of a quadric containing them. We shall say that *five lines lying on a quadric in* [4] *are related*, and prove the following *extension of Pascal's theorem.*

Five lines in [4] *are related if, and only if, on considering them in an arbitrary cyclic arrangement, $a_1a_2a_3a_4a_5$ say, and intersecting each line a_i with the prime $a_{i-1}\,a_{i+1}(i=1, 2, 3, 4, 5, a_0=a_5, a_6=a_1)$, we obtain five points of a prime.**

We shall prove this theorem in §10, by means of algebraic-geometric considerations, after having investigated (in §9) its analytical substratum.† A simple, purely geometric proof of the *necessity* for the stated condition will be indicated in §14.

From §4 we see immediately that:

Five lines $a_1a_2a_3a_4a_5$ in [4] *are related if, and only if, the pairs of primes a_1a_2 and a_3a_4, a_1a_3 and a_4a_2, a_1a_4 and a_2a_3 cut out on a_5 three pairs of points in involution. This property also holds if we interchange the line a_5 with any of the remaining lines.*

9. An identity between two determinants. Let us introduce in [4] homogeneous projective coordinates, and define the line a_i as the join of two distinct points

* We could substitute the prime $a_{i-1}\,a_{i+1}$ in this statement with $a_{i-2}\,a_{i+2}$ (assuming $a_{-1}=a_4$, $a_7=a_2$), since this would simply be tantamount to applying the stated result on considering the five lines in the order $a_1a_3a_5a_2a_4$.

† A proof on the same line can also be given for Pascal's theorem for conics.

$$(1)\qquad P_i(x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}), \qquad Q_i(y_{i1}, y_{i2}, y_{i3}, y_{i4}, y_{i5}) \qquad (i = 1, 2, 3, 4, 5).$$

The general quadric in [4] has the equation

$$\begin{aligned}(2)\qquad & k_{11}x_1^2 + k_{22}x_2^2 + k_{33}x_3^2 + k_{44}x_4^2 + k_{55}x_5^2 \\ & + 2k_{52}x_5x_2 + 2k_{13}x_1x_3 + 2k_{24}x_2x_4 + 2k_{35}x_3x_5 + 2k_{41}x_4x_1 \\ & + 2k_{12}x_1x_2 + 2k_{23}x_2x_3 + 2k_{34}x_3x_4 + 2k_{45}x_4x_5 + 2k_{51}x_5x_1 = 0,\end{aligned}$$

where the k's are 15 parameters not all zero; and we express that it contains the line a_i, by writing the conditions that each of the points (1) lies on (2), and that P_i and Q_i are conjugate with respect to this quadric ($i=1, 2, 3, 4, 5$). This gives altogether 15 linear equations for the k's, whose compatibility condition is

$$(3)\qquad D = \begin{vmatrix} x_{11}^2 & \cdots & 2x_{15}x_{12} & \cdots & 2x_{11}x_{12} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{11}^2 & \cdots & 2y_{15}y_{12} & \cdots & 2y_{11}y_{12} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{11}y_{11} & \cdots & x_{15}y_{12} + x_{12}y_{15} & \cdots & x_{11}y_{12} + x_{12}y_{11} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

In the determinant D we have written explicitly only three rows, arising as indicated from the conditions for the points (1) for $i=1$; the other rows arise likewise in succession for $i=2, 3, 4, 5$. Moreover, the arrangement of the columns in D can be taken to be the one corresponding to that of the terms in (2). The condition (3) is of the third degree in each of the ten sets of variables (1), and gives *the necessary and sufficient condition for the lines $a_1a_2a_3a_4a_5$ to be related.* It is geometrically obvious that the determinant D must be expressible as a cubic form in the Plücker coordinates of these five lines.

We consider next the point

$$(4)\qquad R_i(z_{i1}, z_{i2}, z_{i3}, z_{i4}, z_{i5})$$

intersected on a_i by the prime $a_{i-1}\,a_{i+1}(i=1, 2, 3, 4, 5)$. It is easily seen that R_i has the coordinates

$$(5)\qquad z_{ik} = \begin{vmatrix} x_{i1}y_{ik} - x_{ik}y_{i1} & x_{i2}y_{ik} - x_{ik}y_{i2} & x_{i3}y_{ik} - x_{ik}y_{i3} & x_{i4}y_{ik} - x_{ik}y_{i4} & x_{i5}y_{ik} - x_{ik}y_{i5} \\ x_{i-1,1} & x_{i-1,2} & x_{i-1,3} & x_{i-1,4} & x_{i-1,5} \\ x_{i+1,1} & x_{i+1,2} & x_{i+1,3} & x_{i+1,4} & x_{i+1,5} \\ y_{i-1,1} & y_{i-1,2} & y_{i-1,3} & y_{i-1,4} & y_{i-1,5} \\ y_{i+1,1} & y_{i+1,2} & y_{i+1,3} & y_{i+1,4} & y_{i+1,5} \end{vmatrix},$$

which could also be expressed as trilinear functions of the Plücker coordinates of a_{i-1}, a_i, a_{i+1}. The condition for the five points (4) to be in a prime is

$$\Delta = \begin{vmatrix} z_{11} & z_{12} \cdots z_{15} \\ z_{21} & z_{22} \cdots z_{25} \\ \cdots & \cdots \\ z_{51} & z_{52} \cdots z_{55} \end{vmatrix} = 0, \tag{6}$$

and is clearly of the third degree in each of the ten sets of variables (1).

In §10 we shall establish geometrically the *remarkable identity*

$$D = 32\Delta. \tag{7}$$

This implies that the conditions (3), (6) are equivalent, whence the first theorem of §8 follows immediately.

A purely algebraic proof of (7) seems not to be easy. Hence, in the present paragraph, we confine ourselves to verifying (7) directly for the following numerical values of the variables (1). We assume all these variables to be zero, excepting

$$x_{11} = x_{22} = x_{33} = x_{44} = x_{55} = 1,$$

and

$$y_{15} = y_{12} = y_{21} = y_{23} = y_{32} = y_{34} = y_{43} = y_{45} = y_{54} = y_{51} = 1.$$

Then in the determinant D all the elements of the first five rows are zero, excepting those of the principal diagonal, which are 1; moreover, all the elements of the successive five rows which are not in the first five columns are zero, excepting those of the principal diagonal, which are 2. Hence

$$D = 2^5 \begin{vmatrix} x_{11}y_{12}+x_{12}y_{11} & x_{12}y_{13}+x_{13}y_{12} & x_{13}y_{14}+x_{14}y_{13} & x_{14}y_{15}+x_{15}y_{14} & x_{15}y_{11}+x_{11}y_{15} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{51}y_{52}+x_{52}y_{51} & x_{52}y_{53}+x_{53}y_{52} & x_{53}y_{54}+x_{54}y_{53} & x_{54}y_{55}+x_{55}y_{54} & x_{55}y_{51}+x_{51}y_{55} \end{vmatrix}$$

$$= 2^5 \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = 2^5 \cdot 2.$$

From (5), (6) we deduce at present without difficulty

$$\Delta = \begin{vmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{vmatrix} = 2.$$

The last two equations prove (7) for the indicated numerical values of the variables (1), and show incidentally that *neither* (3) *nor* (6) *holds identically*.

10. Proof of the previously stated results. From §9 we see that, in order to establish (7), we need only to prove that the ratio D/Δ is a constant, *i.e.* is independent of each of the ten sets of variables (1). It is obviously sufficient to demonstrate the independence of D/Δ from one of these sets, *e.g.* from

$$x_{51},\ x_{52},\ x_{53},\ x_{54},\ x_{55}, \tag{8}$$

since the same argument can be applied for the nine remaining sets. This will clearly follow if we prove that the equations (3), (6) represent *the same algebraic variety*, when we interpret the variables (8) as coordinates of a point P_5 variable in [4] and the remaining variables as parameters.

First of all, on fixing generically the sets (1) for $i=1, 2, 3, 4$ we fix the quadruplet of lines $a_1a_2a_3a_4$ in a general position, and so we can consider the complementary quadruplet, $b_1b_2b_3b_4$ say, defined as in §3. Then (3) is a cubic equation for the Plücker coordinates of the line $a_5=O_5P_5$, and represents the cubic complex of lines, K say, defined by the double-four $d=(a_1a_2a_3a_4,\ b_1b_2b_3b_4)$ as in §4. On fixing in (3) also the coordinates $(y_{51}, y_{52}, y_{53}, y_{54}, y_{55})$ of the point O_5, and considering (8) there as coordinates of the variable point P_5, we clearly obtain the equation of the cone of the complex K having O_5 as vertex. As already noticed in §4, this is a cubic cone containing the eight planes projecting d from O_5.

Secondly we remark that, from §9, the equation (6) represents likewise a cubic cone of vertex O_5; and we have to prove that this cone coincides with the previous one. This follows by projection from the final result of §6, if we show that also the cone represented by (6) contains the eight planes projecting d from O_5. We therefore need only to prove that (6) holds, *i.e.* that the five points (4) are in a prime, if a_5 meets one or other of the eight lines of d. We do so now, on distinguishing three cases.

(i) If a_5 meets a_1, then both R_5 and R_1 coincide with the point a_1a_5, so that (6) obviously holds. A similar argument can be applied if a_5 meets a_4.

(ii) If a_5 meets a_2, the prime a_5a_2, and therefore also R_1, is indeterminate. Then (5) shows in fact that $z_{11}=z_{12}=z_{13}=z_{14}=z_{15}=0$, and so (6) holds. A similar argument can be applied if a_5 meets a_3.

(iii) If a_5 meets b_1, the five points (4) lie on the prime a_5a_2. This is obvious for R_5, R_2 which lie on a_5, a_2 respectively, as well as for R_1 which is the intersection of a_1 with this prime. As for R_3, R_4, it is easily seen that these points coincide now with the intersections of b_1 with a_3, a_4 respectively; hence they lie on the prime a_5a_2, since this prime contains the line b_1, which meets both a_5 and a_2. A similar argument shows that, if a_5 meets either b_2, or b_3, or b_4, then the five points (4) lie on the prime a_1a_3, or a_2a_4, or a_3a_5 respectively.

11. On some questions suggested by §3 and §8. The final result of §8 gives a *linear test* for five related lines in [4]. This can be taken as the *definition* of related lines, without the need of mentioning the existence of a quadric contain-

ing the five lines. Another *linear test* is given by the first theorem of §8, which we can restate in a slightly different form as follows:

Let us consider in [4] *a set of five lines arranged cyclically, and suppose that the five points of intersection of each line with the join of its two adjacent lines are in a prime. Then, and only then, the five lines are related, and the same property holds for every cyclical arrangement of the five lines.*

Five related lines thus define an interesting configuration,* which, however, we do not intend to study in detail in this paper. We only remark that the 12 possible cyclical arrangements of the five lines give 12 primes in [4]. The points of intersection of these primes with the five lines are only 30 in number, since each such point lies in two distinct primes. In fact each of the five lines contains 6 of the 30 points, given by its intersections with the primes joining in pairs the remaining four lines, and consisting of three pairs of points in involution (§8). Hence *the configuration defined in* [4] *by five related lines has the characters* $(30_2, 12_5)$.

By duality in [4] we deduce immediately properties of *five related planes*, which for the sake of brevity will not be stated explicitly. Each quintuplet of related lines is transformed into a quintuplet of related planes, by the polarity with respect to the quadric containing its five lines. The relation between two such quintuplets is a reciprocal one, and deserves further investigation.

We now point out a few questions of a different type, also suggested by the results of §3 and §8. We may define the *associated quintuplet* of an arbitrary quintuplet of lines in [4], as the one formed by the lines associated to the five quadruplets of lines contained in the given quintuplet (§2). The new quintuplet of lines has in its turn an associated one, and so on. Thus *sequences of quintuplets* are obtained; and one could investigate those which contain *one or more quintuplets of related lines*, or those (if any) which are *periodic*.

Another interesting problem is that of studying the *sets of* $n>5$ *lines in* [4] *which are five by five related*. A trivial example is offered by n lines in [4] lying on the same quadric.

Part III. Some Applications of Plane and Three-Dimensional Geometry

12. On quintuplets of lines in space. We first prove the following theorem:

Let $a_1a_2a_3a_4a_5$ *be five general lines in ordinary space, meeting a given plane at* $A_1A_2A_3A_4A_5$ *respectively, and denote by* Γ *the conic containing these five points. We construct the point* B_i *in which* a_i *intersects the quadric containing* Γ, a_{i-1}, a_{i+1}, *residually to* $A_i (i=1, 2, 3, 4, 5, a_0=a_5, a_6=a_1)$. *Then the five points* $B_1B_2B_3B_4B_5$ *lie on a quadric passing through* Γ.

* The analogue of the well known configuration of the Hexagrammum Mysticum, defined on a plane by six points of a conic.

The ∞^4 linear system formed by the ordinary quadrics through Γ can be considered, in the well known way, as representative of the prime sections of a quadric Q in [4]. Then $a_1a_2a_3a_4a_5$ represent five lines of [4] lying on Q, and the result stated above is an immediate consequence of the first theorem of §8. Other properties of five lines and a plane in ordinary space, can likewise be deduced from §11.

13. On quintuplets of related circles in space. The ∞^4 linear system formed by the spheres of an ordinary space (*i.e.*, by the ordinary quadrics containing the absolute conic), can be taken as representative of the prime sections of a quadric Q in [4]. It is well known that any two orthogonal spheres thus represent the sections of Q with two primes which are conjugate with respect to Q, and conversely. In this representation, the circles of the ordinary space correspond to the plane sections of Q, since each of the former is the base curve of a pencil of spheres, representing the sections of Q with the primes of a pencil. The sections of Q with a prime and a conjugate plane, are represented in ordinary space by a sphere and a circle orthogonal to it, and conversely; *etc.*, *etc.*

We shall say that *five circles in ordinary space are related*, if the corresponding plane sections of Q lie in five related planes of [4]. This gives a single condition for the five circles. The same condition can obviously be stated otherwise, by saying that the polar lines with respect to Q of the five planes in [4] are related.

From §11 we easily deduce that:

Five circles in ordinary space are related if, and only if, either of the following two properties holds for the five circles taken in a given order, $C_1C_2C_3C_4C_5$ say; then both properties hold for every arrangement of the five circles. (i) *On constructing the sphere through C_i orthogonal to the one orthogonal to C_{i-1} and C_{i+1} ($i=1, 2, 3, 4, 5$, $C_0=C_5$, $C_6=C_1$) we obtain five spheres, and these are orthogonal to another sphere.* (ii) *The centers of the spheres through C_5 orthogonal to the spheres orthogonal to the pairs of circles C_1C_2 and C_3C_4, C_1C_3 and C_4C_2, C_1C_4 and C_2C_3, are three collinear pairs of points in involution.*

14. On quintuplets of points and their polar planes with respect to a null-system. It is well known that the lines of [3] can be represented by the points of a four-dimensional quadric Q^* in [5], in such a way that the lines of a linear complex L in [3] are represented in [5] by the points of the quadric Q section of Q^* with a prime.† Five arbitrary generators of Q are represented in [3] by five arbitrary pencils of lines of L: the centers $A_1A_2A_3A_4A_5$ of these pencils are five arbitrary points of [3], and their respective planes $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ are the polar planes of $A_1A_2A_3A_4A_5$ with respect to L. Finally a prime section of Q is represented in [3] by a congruence of lines of L, *i.e.*, by the totality of the lines of [3] incident with two lines, which are mutually polar with respect to L, and may possibly be infinitely near.

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† This prime does not touch or touches Q^*, according as L is a general or special linear complex.

From the results just recalled and §8, we easily deduce the following theorem:

Let $A_1A_2A_3A_4A_5$ be five general points in ordinary space, and $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ their polar planes with respect to a given (general or special) linear complex of lines. Then the five pairs of planes α_i and $A_{i-1}A_iA_{i+1}$ ($i=1, 2, 3, 4, 5$, $A_0=A_5$, $A_6=A_1$) intersect in five lines having two common transversals.

We obtain an alternative proof of this theorem, by noticing that the five lines indicated are both in the given linear complex, and in the linear complex defined by the conditions of having $A_{i-1}A_iA_{i+1}$ as polar plane of A_i ($i=1, 2, 3, 4, 5$). Hence they are in the linear congruence formed by the common lines of the two complexes.

Conversely, from this proof and the previous hyperspatial interpretation, it is easy to deduce a new proof of the necessity for the condition stated in the first theorem of §8.

15. On quintuplets of oriented elements on a plane. It is well known that S. Lie gave a useful representation between the elements of a plane π and the points of an ordinary space.* This representation was then slightly modified by E. Kasner,† into a birational transformation between the oriented elements of π and the points in space. In this transformation, the oriented circles of π correspond to the lines of a general linear complex L in space; hence two oriented elements of π are co-circular if, and only if, their representative points are conjugate with respect to L. A general line in space corresponds to a *turbine* of π, *i.e.*, to ∞^1 oriented elements whose points form a circle and whose directions are equally inclined to that circle; and conversely. Two lines in space mutually polar with respect to L, lead to two turbines having the same circle as point locus, and whose elements are symmetrically related to the elements of the circle.

Let us now apply the theorem of §14 to the linear complex L. We may consider the line of intersection of α_i and $A_{i-1}A_iA_{i+1}$ as the join of A_i with the point of intersection of α_i and $A_{i-1}A_{i+1}$. From the results recalled above, we then deduce the following theorem.

If five oriented elements $e_1e_2e_3e_4e_5$ of a plane are generally given, we can consider the oriented circle containing e_i and having an element in common with the turbine defined by e_{i-1} and e_{i+1} ($i=1, 2, 3, 4, 5$, $e_0=e_5$, $e_6=e_1$). The five circles thus obtained are isogonal to another circle.

* Geometrie der Berüngstransformationen, I, p. 238.

† The group of turns and slides and the geometry of turbines, Amer. Journ. of Math., 33, 1911, pp. 193–202.

MATHEMATICS IN A LIBERAL EDUCATION*

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1. Introduction. Considerations of the place of mathematics in liberal education have been somewhat neglected during the past three years. The neglect was brought about, of course, by the demand that many men acquire certain mathematical skills in the shortest possible time. Now although the war has brought mathematics before the public eye, it has focussed that eye only on the more immediately useful mathematical skills—skills almost exclusively manipulative in nature. The public may well tend to lose sight of an entirely different kind of skill in mathematics. This skill, which has values in liberal education, is one that the public had already begun to challenge two decades before the war.

2. Some shortcomings and misconceptions. As I reflect on what I have read in the past few years about the place of mathematics in liberal education, and about mathematics curricula designed for such education, I am impressed by several things. One of these is that it is difficult to discover a definition of "liberal education" in many of the papers which use the term. Possibly the writers regard the concept as too simple, or too general, to admit of definition. A number of them imply that for them a liberal education is one which enables the student to solve certain specific problems he will presumably meet later in his vocational or social life. This sort of thing is not education, but a kind of training which may have a place in a school designed to train, but which has little place in a school devoted to the liberal arts.

Then there are writers who discuss mathematics as a discipline, and so come somewhat closer to the point. Many of them are concerned with the problem of "transfer." Here I am impressed by the number of writers who, unable to find any significant transfer of mathematical methods to non-mathematical situations, tend to overlook, among other things, the kind of mathematical instruction received by the students they are trying to test. In this connection they disregard not only the content of such instruction, but its method of presentation as well. I am also impressed by the fact that only a very small number of writers are bold enough to question whether or not the objectives of mathematics in liberal education have yet been clearly stated; or, if stated, whether or not they have yet been measured by means of tests and examinations. Of this small group of critics, some believe that significant tests will eventually be found, but have not been found to date. A very few suspect that significant tests may never be found.

Leaving the question of tests to the examiners, I should like to discuss briefly the general objectives of liberal education, to suggest an appropriate place for mathematics in such an education, and to describe a course designed to fill that place.

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* Address presented at the annual meeting of the Mathematical Association of America, Chicago, Illinois, November 26, 1944.

3. Liberal education and mathematics. A liberal education, properly speaking, is one which liberates the student's mind. It does so by providing him with intellectual disciplines of various kinds. He must be taught not only to read and write, but to analyze and interpret what he has read, to look for premises and conclusions of arguments, to recognize them when he has found them, and to discover the presuppositions which lead to the particular choice of the premises used. He must become acquainted not only with a part of humanity's store of knowledge, but with the various methods by which knowledge of different kinds is gained, with methods by which premises are formulated, with methods by which premises lead to conclusions, and with methods by which conclusions are validated. No single discipline can do all of these things. Nor is it correct to assume that there exists a one-to-one correspondence between fields of knowledge and disciplines appropriate to them. There is a variety of disciplines, some of which are appropriate to one field and some to another field, with much overlapping among them.

Now some of these disciplines involve the use of ordinary numerical reckoning. The value of mathematics in this very narrow sense is almost too obvious to mention. But consider mathematics as a discipline in itself—that is to say, as a body of concepts and methods which constitute a way of thinking. Surely mathematics is such a discipline. It deals almost exclusively with premises and conclusions, and with deductive reasoning, which is one of the more important methods of drawing conclusions from premises. Moreover, clarity and precision of definitions and assumptions, and rigor in reasoning, can be more nearly attained and more simply studied in mathematics than in the other disciplines. Is not this the real place of mathematics in a liberal education—not simply as a subject matter, or as a discipline applicable only to its own subject matter, but as a discipline which is applicable to almost every intellectual activity of man?

Mathematics was given without question the place it deserves in liberal education until the last two or three decades. In 1928 Florian Cajori published a compilation of the opinions of all writers known to him, an historian of mathematics, who had expressed themselves on the relation of mathematics to liberal education.* The list includes the names of over seven hundred men, ranging in time from the Greek period to the twentieth century, and ranging in professions from philosophers, mathematicians, and scientists to literary men, statesmen, business men, and lawyers. Over four-fifths of these men placed a high value on mathematics as a part of liberal education. Although this accumulation of opinion may not constitute public opinion, it is certainly representative of it to some degree. At any rate, it is doubtful whether we shall have public opinion so strongly on our side in the years which lie immediately ahead. Returned veterans, for example, having been subjected to accelerated courses in mathematics, often at the hands of inexperienced teachers, will know little or nothing of the value of mathematics as an appropriate study for all students, but will regard

* Mathematics in Liberal Education, Boston, Christopher, 1928.

mathematics as a study relevant only to vocations in which they are no longer engaged. Perhaps our best argument will lie in mathematics courses truly—and thoughtfully—designed for all students, regardless of the occupation or profession they may expect to enter.

4. A course designed for liberal education. Since the autumn of 1943 such a course has been offered in the College of the University of Chicago. It should perhaps be pointed out that Chicago differs from most colleges in that students are accepted after they have completed two years of high school. Many of you, however, are also faced with entering students who have had no more than two years of high school mathematics, and what I have to say here applies equally well to them. Obvious modifications of the program for students who enter college with more than two years of mathematics will no doubt occur to you.

Our general course, then, is a one-year course (two semesters, or three quarters), meeting five times per week, and presupposing a knowledge of elementary algebra and plane geometry such as is ordinarily acquired in one-year high school courses in these subjects. Note the expression, "such as is ordinarily acquired . . . ." Rather than bring up the time-worn question of pre-college training in mathematics, we might as well resign ourselves to accepting students who have been exposed to a year of algebra and a year of geometry. Actually, much can be done with such students even though they may have failed to catch that to which they were exposed.

The work of the course falls into four parts: (1) logical structure, (2) geometry, (3) algebra, and (4) coördinate geometry. Of these four parts, the first two together, the third, and the fourth, each occupy about one third of the time devoted to the course.*

Part 1 (logical structure) consists of a general consideration of the role played by definitions, assumptions, and methods of reasoning in fields of thought in general, and in mathematical and scientific fields of thought in particular. Here the student is made aware of the necessity for undefined terms and assumptions. He begins to learn something about the importance of clear and precise formulations of definitions, assumptions, and other propositions. He acquires an elementary understanding of the notions of consistency and independence of postulates. He suddenly realizes that in order to be able to demonstrate anything, he must accept some fundamental logical structure by which to reason. He begins to see what a proof *is*, and how the postulates of contradiction and excluded middle enter into indirect proof. And in his study of relations between propositions, he is led to understand the significance of converses, inverses, contrapositives, contradictories, and necessary and sufficient conditions. Needless to say, the ideas developed here constitute the fundamental framework of the entire course.

Part 2 (geometry) consists of a brief critical examination of Euclid's definitions and postulates, a brief review of plane geometry, and an extension of plane

* The text for the course, prepared by the author of the present article, is Fundamental Mathematics, Chicago, University of Chicago Bookstore, 1944–45.

geometry to certain portions of euclidean space geometry. The postulates used in the review of plane geometry are postulates formulated by the late G. D. Birkhoff.* Based on scale and protractor, they are different from, but equivalent to, the ordinary euclidean postulates. They serve to give the student practice in reasoning from a set of postulates with which most students are unfamiliar. They make it possible to develop, in some ten or twelve theorems, nearly everything of importance to the course concerning rectilinear figures in the plane (Pythagorean theorem, theorems on similarity, angle sums, parallelism and perpendicularity, *etc.*). And resting ultimately upon the properties of real numbers, they lead naturally to the work in algebra and lay the foundation for the work in coördinate geometry. The discussion of euclidean space geometry is designed to show the student how an additional assumption or two is sufficient for the purposes of extending geometry from two dimensions to three, and to give him some understanding of the relations of lines and planes in space.

Part 3 (algebra) consists of a postulational development of certain portions of algebra, including elementary algebra and parts of "intermediate" and "college" algebra. This approach not only serves to bring order into what the student looks upon as a vast collection of dissociated facts and techniques, but offers an opportunity for badly needed remedial work in connection with those techniques. The work here centers around a relatively rigorous development of number systems, which in turn leads to a relatively thorough investigation of variables and functions.

Starting with the natural numbers, these numbers and addition are accepted as undefined concepts. Multiplication is defined in terms of addition, and the ordinary closure, commutative, associative, and distributive postulates are formulated. Subtraction is introduced as the inverse of addition, and the natural numbers extended to the integers through postulation of identity and inverse elements for addition. Division is introduced as the inverse of multiplication, and the integers extended to the rational numbers through postulation of an inverse element for multiplication. Evolution, or root extraction, is introduced as the inverse of involution, or raising to a power, and the necessity for further generalizations of the number system is made clear. Here rigor tends to break down, for the obvious reason that this elementary course is not a course in real variables. Nevertheless, real numbers are introduced by means of the Dedekind cut, a concept which is by no means as difficult for the student to grasp as it is generally thought to be. This contention is particularly true in the case of the student whose geometry is founded on the notion of one-to-one correspondences between points of lines and members of classes of numbers. No attempt is made, however, to develop the real numbers in any complete sense. A return to rigor is accomplished in the extension of real numbers to complex numbers through

* "A set of postulates for plane geometry, based on scale and protractor," Annals of Mathematics, series 2, vol. 33, 1932, pp. 329–345.

postulation of an imaginary unit. Theorems on special products and factoring are introduced in connection with integers, theorems on order in connection with rational numbers, and theorems on exponents in connection with real numbers. The study of number systems closes with an investigation of the Peano postulates, by means of which the natural numbers and addition may be defined, and the original postulates for natural numbers proved. Particular emphasis is laid on Peano's fifth postulate—that of mathematical induction—as a new method of proof.

The second portion of part 3 opens with a discussion of the role played by functional relations in the sciences. Variables and functions are introduced with care, and a wide variety of functions is examined. The material also includes the study of functions of several variables, the construction of functional relations, the introduction of functional notation, and various classifications of functions. Investigation of zeros of functions leads to the general question of the solution of equations, which is treated in detail through polynomials of the second degree. A short time is also spent in the study of functions of positive integral variables: arithmetic and geometric series, and permutations and combinations, together with their applications to probability and the binomial theorem.

Part 4 (coördinate geometry) is the logical outgrowth of parts 2 and 3, the student being well prepared for this union of geometry and algebra. For in geometry he was introduced to the assumption of a one-to-one correspondence between the points of a line and the real numbers, and in the discussion of Dedekind cuts he was able to gain a clearer notion of the implications of such an assumption. Linear coördinates are thoroughly investigated before rectangular coördinates are introduced. Plane rectangular coördinate geometry is then developed in detail through the straight line and the circle, and the graphs of algebraic functions are studied in a general way. More than usual attention is paid to the complete correspondence between geometric concepts, relations, and operations on the one hand, and algebraic concepts, relations, and operations on the other. Considerations of the circle lead naturally to a discussion of the circular functions, their properties, and their applications. Thus the fundamentals of trigonometry are treated, not as a separate subject, but as an appropriate part of the general study of functions and functional behavior.

5. Emphasis on method of presentation. My point in describing this course designed for liberal education is not only to argue for the inclusion in such a course of the particular material cited, or at least for the inclusion of the four parts named, but to argue for the particular method of presentation used. Indeed, the pedagogy is to be stressed far more than the content.

For example, the work of the first part of the course—logical structure—does not end when the second part is begun, but continues on, transformed from something *thought about* to something *used in thinking*.

Again, throughout the course emphasis is laid on the cultivation of an acute critical attitude on the part of the student. He is held to a level of rigor higher

than that which most students of his age are credited with the ability to appreciate. He is not often faced with the lame phrase, "it can be shown that," and when he is faced with it, an attempt is made to show him why.

Emphasis is also laid on the cultivation of skill in fundamental manipulative techniques. This emphasis is not only consistent with, but actually necessary to, the objectives of the course. Full appreciation of mathematics is impossible without such skill, and it is evidently of help to the student in his general courses in the sciences. On the other hand, this emphasis is confined to the more fundamental techniques, and is not allowed to run away with the course.

The students' attention is directed to relevant applications of his work in the other sciences, but no attempt is made to force such references. So-called practical applications of mathematics are generally included in courses for one of two purposes: to train the student for particular vocational work, or to motivate his interest. The first of these purposes is not one of the purposes of a liberal education, and the second is unnecessary in the course described above if it is properly presented.

It should be added that this is actually the first of two general courses in mathematics offered by the College of the University of Chicago, and that the College faculty has voted to make it a required part of their program of liberal education. A second course—an optional one, designed for students who wish to continue their work in mathematics—is in the process of construction. This second course is of necessity more technical than the first, in that it aims to prepare the student for more specialized work in mathematics and science. On the other hand, the method of presentation used in the first course is carried over into the second.

6. Conclusion. The first course, which has been described in detail, is a proposed course in mathematics for purposes of liberal education. I believe that such a course, properly taught, is more appropriate than most conventional courses for the student who plans to study mathematics only one year beyond algebra and geometry. In this I think many of you may agree with me. I wonder how many of you would also agree with me if I were to make the far more radical suggestion that such a course, properly taught, is also more appropriate than most conventional courses at the same level for the student who plans to continue his work in mathematics or science. My belief that this is the case is based on the following grounds: that an order of learning in which first emphasis is placed on the mastery of a conception of mathematical systems and of skill in following their development makes mastery of subsequently studied systems more rapid and more complete. Whereas to place first emphasis on the mastery of a large body of mathematical formulae and of skill in their manipulation makes mastery of subsequent formulae and manipulation almost as difficult as mastery of the first, and makes more and more difficult any later attempts to master and appreciate mathematical systems as such.

THE VALUATION OF RISKS

W. J. MAYS, The Volunteer State Life Insurance Company

1. Introduction. The purpose of this paper is to develop some of the ideas set forth by Laplace in the chapters of his "Theorie Analytique des Probabilités" which treat upon *l'esperance moral*, or the moral expectation that may be assigned to a gain or loss contingent upon the outcome of a doubtful event. These ideas are summarized in the following quotation:*

"The probability of events serves to determine the hopes and fears of persons interested in the outcomes of the events. The word 'expectation' has various acceptations; it expresses generally the advantage to one who expects some fortune merely on basis of a likely supposition. In the theory of risks, that advantage is the product of the sum expected by the probability of its being obtained; it is the partial sum which ought to be exchanged when one cannot run the risk of the outcome, on the supposition that the distribution of the whole sum is made in proportion to the probabilities. This is the only equitable means of distribution when every foreign circumstance is removed, for those who have equal degrees of probability have equal rights to the sum expected. We call this advantage mathematical expectation to distinguish it from moral expectation, which depends, as the former, upon the value expected and upon the probability of its realization, but which is governed by a thousand circumstances that are nearly always impossible to define, and even more so to subject to calculation. Although these circumstances increase or diminish the value expected, the moral expectation itself may be considered as the product of the value by the probability of obtaining it; but then a distinction must be made for the value expected, between its relative value and its absolute value; the latter is independent of the motives which make it desirable, whereas the former increases with such motives.

"No general rule can be given for appraising the relative value; however, it is natural to suppose the relative value of an infinitely small sum to be in direct proportion to its absolute value and in inverse proportion to the fortune of the interested person. Indeed, it is clear that a franc has very little value for one who has a large number of francs and that the most natural method of estimating its relative value is to suppose such value to be in inverse proportion to the number (of francs in the fortune)."

2. Bernoulli's Hypothesis. The last paragraph in the preceding quotation contains a statement of Bernoulli's Hypothesis. Symbolically, if dL represents the infinitesimal relative moral value of an infinitesimal gain of absolute value dA to a person whose fortune† is of absolute value A, Bernoulli's Hypothesis may be stated in the form

* Théorie Analytique des Probabilités, Book II, Chapter 2.

† The term "fortune" as used in this paper is intended to mean that portion of a person's means which is available for speculation, or otherwise properly subject to risk.

$$dL = k\frac{dA}{A} \tag{1}$$

where k is a constant of proportionality.

By integration of (1) we find the relative moral value L of a finite gain to be

$$L = k \log A + c \tag{2}$$

where c is a constant of integration and A is taken to include the gain. To find c, notice that if L is zero, $c = -k \log A_0$, where A_0 is the initial amount of the fortune, that is, the amount before inclusion of any gain. Equation (2) may now be written

$$L = k \log\left(1 + \frac{X}{A_0}\right) \tag{3}$$

where X is the absolute value of the gain.

Suppose that one possessed initially of a fortune A has a probability p_1, of realizing a sum P_1, a probability p_2 of realizing a sum P_2, $\cdots$, a probability p_n of realizing a sum P_n, the probabilities being mutually exclusive. Then a value X of his expectation is determined, on basis of Bernoulli's Hypothesis, from the equation

$$\log\left(1 + \frac{X}{A}\right) = p_1 \cdot \log\left(1 + \frac{P_1}{A}\right) + p_2 \cdot \log\left(1 + \frac{P_2}{A}\right) + \cdots + p_n \cdot \log\left(1 + \frac{P_n}{A}\right), \tag{4}$$

$$\text{or } \frac{X}{A} = \left(1 + \frac{P_1}{A}\right)^{p_1} \cdot \left(1 + \frac{P_2}{A}\right)^{p_2} \cdots \left(1 + \frac{P_n}{A}\right)^{p_n} - 1. \tag{5}$$

The value, X, of the expectation as determined from equation (4) or (5) has been called the physical value of the expectation to distinguish it from the relative moral value and from the mathematical value. The true significance of L, the relative moral value, is that it represents not particularly a value but rather a transformation function which, when assigned, serves to determine the physical value, X, of the expectation. Although the physical value is dependent upon the particular transformation function employed (the logarithmic function, for Bernoulli's Hypothesis), it is measured in the same scale as the absolute values of the fortune and the gains. All that has been said about gains applies similarly to losses provided appropriate changes in signs are made (that is, if P is the amount of a loss, represent the loss by $-P$). It is, of course, assumed that no loss is to be considered which would be in excess of the fortune.

Suppose now that one possessed of a fortune A engages in a play for which there is a probability p of winning a sum P and a probability q of losing a sum Q

such that $pP=qQ$ and $p+q=1$, or in other language a play that is fair in the mathematical sense. Then, on basis of Bernoulli's Hypothesis

$$1+\frac{X}{A}=\left(1+\frac{P}{A}\right)^{p}\cdot\left(1-\frac{Q}{A}\right)^{q}. \tag{6}$$

Now, it is easy* to show that the right-hand member of (6) is less than unity; hence, on basis of Bernoulli's Hypothesis, it is disadvantageous for one of limited means to play a "fair" game of chance.

There is an interesting interpretation of equation (6).

If one repeats the play a large number of times, say n times, he will win, on the average, pn times and will lose qn times. Now, if at each play the amounts at risk are so changed that they are always in proportion to his changing fortune, then each time he wins he will multiply his fortune by $(1+P/A)$ and each time he loses he will multiply his fortune by $(1-Q/A)$. The geometric mean factor for the change in fortune at each play is $1+X/A$ (X being negative), which is the constant factor by which he may certainly multiply his fortune n times to arrive at the same end result as if he played n times. The role of the logarithmic transformation is to transform constant values of the amounts at risk into values that are always proportional to the player's changing fortune.

3. A general hypothesis. The results obtained thus far have depended upon the choice of the logarithmic function for the moral value. Evidently, a considerably more general treatment of the subject of moral values is possible, if we use the following hypothesis:

The physical value, X, of gaining (or losing) P_1, with probability p_1, P_2, with probability p_2, $\cdots$, P_n with probability p_n (the probabilities being mutually exclusive) for one whose fortune is initially A, is determined from the relation

$$f(A+X)=p_1f(A+P_1)+p_2f(A+P_2)+\cdots+p_nf(A+P_n), \tag{7}$$

where f is an arbitrary function except for the following restrictions which are imposed from practical considerations

(i) f is a continuous function of all the variables over their ranges.

(ii) For each pair of values of A, $A_1>A_2$, the corresponding physical values X_1 and X_2, determined from (7) for a fixed mathematical expectation of gain, must satisfy the inequality $X_1>X_2$, and for a fixed mathematical expectation of loss, must satisfy the inequality $X_1<X_2$.

(iii) As A is indefinitely increased, the function f must be such that equation (7) approaches a limiting form and the value of X determined therefrom must approach the mathematical expectation, that is,

$$\lim_{A\to\infty} X=p_1P_1+p_2P_2+\cdots+p_nP_n. \tag{8}$$

A general function that satisfies these restrictions is $(A+X)^m$, where m is

* The geometric mean of the quantities $(1+P/A)$ and $(1-Q/A)$ weighted by p and q, respectively, is less than the correspondingly weighted arithmetic mean, which is unity.

less than unity. If m is zero, the result is indeterminate; in consequence, to obtain continuity, let $\log (A+X)$ be used for the function corresponding to zero value of m.

For a concrete illustration of the hypothesis, consider a person whose fortune is \$10,000, of which \$5,000 is invested in a home. Suppose that the probability of the destruction of his home by fire in a given period of time is 1/100. The physical values of his expectation of loss are shown for several transformation functions in the following table:

Formula: $$f(10{,}000 - X) = \frac{1}{100} \cdot f(5{,}000) + \frac{99}{100} \cdot f(10{,}000)$$

$f(x)$	X (physical value of loss)
x	\$50·00 (mathematical expectation)
$x^{.8}$	53·23
$x^{.6}$	56·64
$x^{.4}$	60·45
$x^{.2}$	64·60
$\log x$	69·07
x^{-1}	99·01

It appears that, if moral valuation is admissible in accordance with the hypothesis, there is a margin between the mathematical expectation and the physical value in which an insuring company (whose resources are large in comparison with the fortune of the individual) may assign a premium that will be profitable to itself and at the same time advantageous to the insured. The hypothesis is thus in agreement with universal experience on this point.

4. The St. Petersburg problem. This classic problem might well be used to introduce the whole subject of moral values. It may be stated as follows:

A person is offered an expectation of winning \$1 if a coin turns up heads on the first throw, \$2 if it turns up heads on the second throw but not before, \$4 if it turns up heads on the third throw but not before, and so on, *ad infinitum*, the prize doubling for each throw. What price may he judiciously pay for this?

The following table shows the physical values of the expectation to one whose fortune is \$100 on the basis of several transformation functions.

Formula: $$f(100) = 1/2f(101 - X) + 1/4f(102 - X) + 1/8f(104 - X) + 1/16f(108 - X) + \cdots$$

Transformation Function $f(x)$	Physical Value of Expectation X
x	∞ (same as mathematical expectation)
$x^{1/2}$	\$5·32
$\log x$	4·35
x^{-1}	3·67
x^{-2}	3·35

5. Subdivision of risks. A question of some interest arises in connection with the subdivision of risks. As before let p be the probability of gaining (or losing) a sum P, with respect to a person whose fortune is initially A. We wish to inquire about the moral advantage of subdividing the single risk into many risks, such that the total mathematical expectation of the subdivided risks is the same as the mathematical expectation of the single risk. The subdivided risks are assumed not to be mutually exclusive.

We shall investigate two particular cases of this general problem.

Case I. Probabilities of subdivided risks:

$$\pi_1 = \pi_2 = \pi_3 = \cdots = \pi_n = p.$$

Amounts of subdivided risks:

$$\Pi_1 = \Pi_2 = \Pi_3 = \cdots = \Pi_n = \frac{P}{n}.$$

If X represents the physical value of all expectations from the subdivided risks, we have

$$\begin{aligned} f(A + X) = {} & p^n \cdot f\left(A + \frac{nP}{n}\right) + np^{(n-1)} \cdot (1 - p) \cdot f\left(A + \frac{n-1}{n} P\right) \\ & + \frac{n(n-1)}{2} \cdot p^{(n-2)} \, (1 - p)^2 \cdot f\left(A + \frac{n-2}{n} P\right) + \cdots \\ & + \frac{n(n-1)}{2} \cdot p^2 \cdot (1 - p)^{(n-2)} \cdot f\left(A + \frac{2}{n} \cdot P\right) \\ & + np \cdot (1 - p)^{(n-1)} \cdot f\left(A + \frac{P}{n}\right) + (1 - p)^n \cdot f(A). \end{aligned} \tag{9}$$

By suitably increasing n, we can concentrate the important terms of the series on the right into as narrow range (with reference to the order of magnitude of P) as we please about the term

$$\frac{n!}{(pn)!(n - pn)!} \cdot p^{pn} \cdot (1 - p)^{n-pn} \cdot f\left(A + \frac{pn}{n} \cdot P\right).$$

Whence as n is indefinitely increased, the limit of the sum of the series becomes $f(A + pP)$ and the physical value, X, of the expectation from infinite subdivision is pP, which is the same as the mathematical expectation. This result calls to mind the adage about not putting all one's eggs in the same basket.

Case II. Probabilities of subdivided risks:

$$\pi_1 = \pi_2 = \pi_3 = \cdots = \pi_n = \frac{p}{n}.$$

Amounts of subdivided risks:

$$\Pi_1 = \Pi_2 = \Pi_3 = \cdots = \Pi_n = P.$$

The equation from which the physical value of all expectations is determined is:

$$\begin{aligned} f(A+X) = {} & \left(1-\frac{p}{n}\right)^n \cdot f(A) + \frac{p}{n}\left(1-\frac{p}{n}\right)^{n-1} \cdot n \cdot f(A+P) \\ & + \left(\frac{p}{n}\right)^2 \cdot \left(1-\frac{p}{n}\right)^{n-2} \cdot \frac{n(n-1)}{2} \cdot f(A+2P) + \cdots \\ & + \left(\frac{p}{n}\right)^{n-2} \cdot \left(1-\frac{p}{n}\right)^2 \cdot \frac{n(n-1)}{2} \cdot f(A+\overline{n-2}\,P) \\ & + \left(\frac{p}{n}\right)^{n-1} \cdot \left(1-\frac{p}{n}\right) \cdot n \cdot f(A+\overline{n-1}\,P) + \left(\frac{p}{n}\right)^n \cdot f(A+nP). \end{aligned} \tag{10}$$

As n is indefinitely increased, the equation becomes in the limit

$$\begin{aligned} f(A+X) = e^{-p}\Big\{ & f(A) + p \cdot f(A+P) + \frac{p^2}{2!} \cdot f(A+2P) + \cdots \\ & + \frac{p^k}{k!} \cdot f(A+kP) + \cdots \Big\}. \end{aligned} \tag{11}$$

If $P = \frac{1}{2}A$, $p = 1$, and f is the cube root of the argument, then X, as obtained from equation (11) is approximately $.449A$. In general, the moral value of the expectation of gain from this mode of subdivision is less than the corresponding mathematical expectation.

6. Moral disadvantage from possibility of exhaustion of funds in repeated plays. In the quotation at the beginning of this paper Laplace says that moral values are dependent upon a thousand foreign circumstances that are very difficult, if not impossible, to subject to calculation. At least one of these circumstances is amenable to fairly rigorous treatment, provided that a reasonable assumption is made. We refer to the disadvantage to one possessed of limited means that he may exhaust his entire fortune at some stage of a game in which a play is repeated many times. The assumption that will be made is that this disadvantage may be reasonably compensated by adjustment of the probabilities for a play so that the person of limited means may, likely as not, continue the game indefinitely with an opponent of infinite means.

Suppose that two players A and B, possessed initially of fortunes M and N respectively, engage in game consisting of repetitions of the same play. For each play A has a probability p of winning a sum P and B has a probability q of winning a sum Q such that $p+q=1$. The game is to continue until one or the other of the players loses his entire fortune. We require the probability that A will win.

Let U_x represent A's probability of wining the game at a stage when A's fortune is x. Set up the difference equation

$$U_x = pU_{x+P} + qU_{x-Q}. \tag{12}$$

If the indicial equation

$$r^{P+Q} - \frac{1}{p} \cdot r^Q + \frac{1-p}{p} = 0 \tag{13}$$

has no double roots, the solution of the difference equation (12) is

$$U_x = C_1 \cdot r_1^x + C_2 \cdot r_2^x + C_3 \cdot r_3^x + \cdots + C_{P+Q} \cdot r_{P+Q}^x, \tag{14}$$

where $r_1, r_2, r_3, \cdots, r_{P+Q}$ are the roots of the indicial equation and $C_1, C_2, C_3, \cdots, C_{P+Q}$ are constants determined from the $(P+Q)$ conditions:

$$\begin{cases} C_1 + C_2 + C_3 + \cdots + C_{P+Q} = 0 \\ C_1 \cdot r_1 + C_2 \cdot r_2 + C_3 \cdot r_3 + \cdots + C_{P+Q} \cdot r_{P+Q} = 0 \\ C_1 \cdot r_1^2 + C_2 \cdot r_2^2 + C_3 \cdot r_3^2 + \cdots + C_{P+Q} \cdot r_{P+Q}^2 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_1 \cdot r_1^{Q-1} + C_2 r_2^{Q-1} + C_3 \cdot r_3^{Q-1} + C_{P+Q} \cdot r_{P+Q}^{Q-1} = 0 \\ C_1 \cdot r_1^{(M+N-P+1)} + C_2 \cdot r_2^{(M+N-P+1)} + \cdots + C_{P+Q} \cdot r_{P+Q}^{(M+N-P+1)} = 1 \\ C_1 \cdot r_1^{(M+N-P+2)} + C_2 \cdot r_2^{(M+N-P+2)} + \cdots + C_{P+Q} \cdot r_{P+Q}^{(M+N-P+2)} = 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_1 \cdot r_1^{(M+N)} + C_2 \cdot r_2^{(M+N)} + \cdots + C_{P+Q} \cdot r_{P+Q}^{(M+N)} = 1 \end{cases} \tag{15}$$

If each play is fair in the mathematical sense, that is, if $pP = qQ$, the indicial equation becomes

$$r^{(P+Q)} - \frac{P+Q}{Q} \cdot r^Q + \frac{P}{Q} = 0, \tag{16}$$

which has a double root for $r = 1$. The solution of the difference equation for this case is

$$U_x = C_1 + C_2 \cdot x + C_3 \cdot r_3^x + \cdots + C_{P+Q} \cdot r_{P+Q}^x, \tag{17}$$

where $r_3, \cdots, r_{P+Q}$ are the simple roots of the indicial equation other than unity, and the constants are to be determined from the $P+Q$ conditions

$$\begin{aligned} C_1 + C_2 \cdot x + C_3 \cdot r_3^x + \cdots + C_{P+Q} \cdot r_{P+Q}^x = 0 \\ \text{for } x = 0, 1, 2, \cdots, Q-1 \\ C_1 + C_2 \cdot x + C_3 \cdot r_3^x + \cdots + C_{P+Q} \cdot r_{P+Q}^x = 1 \\ \text{for } x = (M+N-P+1), (M+N-P+2), \cdots, (M+N). \end{aligned} \tag{18}$$

By Pellet's Theorem,* $Q-1$ of the simple roots are of absolute value less

* See Lewis Weisner's paper entitled "Moduli of the roots of polynomials and power series" and reference, page 33, January 1941 issue of this MONTHLY.

than unity. Suppose now that the sum of the fortunes of A and B $(M+N)$ is relatively large in comparison with the amounts at risk for each play, P and Q. Then it can be shown that for large values of x

$$U_x \cong C_1 + C_2 x. \tag{19}$$

For, as x increases, all terms involving roots of absolute value less than unity must approach zero. If any term involving a root of absolute value greater than unity were to contribute finite value to the probability for some value of x, then a larger value of x could be found for which the value of the term would be so large that it would produce an impossible value for the probability U_x. We conclude that when $(M+N)$ becomes very large, all terms of the form $C_i r_i^x$ in which r_i is not unity tend to vanish.

A satisfactory approximation to the values of the coefficients C_1 and C_2 when $(M+N)$ is large may be obtained by solving the $Q+1$ equations.

$$\begin{cases} C_1 + C_{i_1} + C_{i_2} + \cdots + C_{i_{Q-1}} = 0 \\ C_1 + C_2 + C_{i_1} \cdot r_{i_1} + C_{i_2} \cdot r_{i_2} + \cdots + C_{i_{Q-1}} \cdot r_{i_{Q-1}} = 0 \\ C_1 + 2C_2 + C_{i_1} \cdot r_{i_1}^2 + C_{i_2} \cdot r_{i_2}^2 + \cdots + C_{i_{Q-1}} \cdot r_{i_{Q-1}}^2 = 0 \\ \cdot \\ \cdot \\ \cdot \\ C_1 + (Q-1) \cdot C_2 + C_{i_1} \cdot r_{i_1}^{Q-1} + C_{i_2} \cdot r_{i_2}^{Q-1} + \cdots + C_{i_{Q-1}} \cdot r_{i_{Q-1}}^{Q-1} = 0 \\ C_1 + (M+N) \cdot C_2 = 1 \end{cases} \tag{20}$$

where $r_{i_1}, r_{i_2}, r_{i_3}, \cdots, r_{i_{Q-1}}$ are the $Q-1$ roots whose absolute values are less than unity.

The values of C_1 and C_2 thus found yield

$$U_x \cong \frac{-\lambda}{M+N-\lambda} + \frac{x}{M+N-\lambda} \quad \text{where}$$
$$\lambda = \frac{1}{1-r_{i_1}} + \frac{1}{1-r_{i_2}} + \cdots + \frac{1}{1-r_{i_{Q-1}}}. \tag{21}$$

If both x and $M+N$ are very large in comparison to λ, we have $U_x = x/\overline{M+N}$; that is, either player's chance of winning at any stage of the game is the ratio of his fortune at that stage to the sum of his fortune and his opponent's fortune. In particular, A's chance of winning the game in the beginning is approximately $M/\overline{M+N}$.

Suppose now that the probabilities for a play are varied slightly in A's favor, but that the amounts at risk are the same as before. Suppose further that A's fortune is large in comparison to the amounts at risk for each play but small in comparison to his opponent's fortune. Let us find A's chance of winning at a stage of the game when his fortune is x.

The roots of the indicial equation (13) are now all distinct. One root is unity; another root is less than unity by as small amount as we please, depending upon

how little we vary the probabilities for a play in A's favor; $(Q-1)$ of the roots have absolute values less than unity by finite amounts; and the remaining roots have absolute values greater than unity by finite amounts. Denote by $(1-\Delta)$ the root that approaches unity as the probabilities for a play approach mathematically "fair" values.

Then, the solution of the difference equation is approximately for large values of x

$$U_x \cong C_1 + C_2(1-\Delta)^x \tag{22}$$

where C_1 and C_2 are constants to be determined from the $Q+1$ conditions

$$\begin{aligned} &C_1 + C_2\cdot(1-\Delta)^x + C_{i_1}\cdot r_{i_1}^x + C_{i_2}\cdot r_{i_2}^x + \cdots + C_{i_{Q-1}}\cdot r_{i_{Q-1}}^x = 0; \\ &\qquad\qquad x = 0, 1, 2, 3, \cdots, Q-1, \\ &C_1 + C_2\cdot(1-\Delta)^{M+N} = 1; \end{aligned} \tag{23}$$

$r_{i_1}, r_{i_2}, \cdots, r_{i_{Q-1}}$ denoting as before the roots of absolute value less than unity. The values of C_1 and C_2 thus obtained are

$$C_1 = \frac{1}{1-(1-\Delta)^{M+N}\cdot\dfrac{(1-r_{i_1})(1-r_{i_2})\cdots(1-r_{i_{Q-1}})}{(1-\Delta-r_{i_1})(1-\Delta-r_{i_2})\cdots(1-\Delta-r_{i_{Q-1}})}}$$

and

$$C_2 = -C_1\cdot\frac{(1-r_{i_1})(1-r_{i_2})\cdots(1-r_{i_{Q-1}})}{(1-\Delta-r_{i_1})(1-\Delta-r_{i_2})\cdots(1-\Delta-r_{i_{Q-1}})}.$$

If B's fortune is infinite and if Δ is very small but not vanishingly small, we have approximately for A's probability U_x of winning the game at a stage when his fortune is x

$$U_x \cong 1-(1-\Delta)^x. \tag{24}$$

From equation (24) let us determine how much the probabilities for each play must be varied in A's favor, when A is possessed initially of a fortune of large but finite amount M, in order that A may, likely as not, continue the game indefinitely with an opponent whose fortune is infinite. We have

$$(1-\Delta)^M = \tfrac{1}{2}, \quad \text{and} \quad \Delta \cong \frac{\log 2}{M}.$$

From equation (13), if we assume that Δ is very small, we obtain for the corresponding increase δp in the probability of A's winning a play

$$\delta p \cong \frac{PQ}{2(P+Q)}\cdot\Delta. \tag{25}$$

Upon substituting the value previously found for Δ, we have

$$\delta p \cong \frac{\log 2}{2} \cdot \frac{PQ}{(P+Q)M} \tag{26}$$

for the increase in A's probability of winning a play over the mathematically "fair" probability, other quantities remaining the same, to place A in position, likely as not, to continue the game indefinitely. We shall refer to the increase, δp, as A's margin of probability. Note that if the amounts at risk are proportionately increased, then A's margin must be increased in direct proportion, in order that he may maintain his advantage.

Let us now find the particular transformation function that produces the moral advantage corresponding to A's margin of probability. Using the same symbols as in the paragraph immediately preceding, we have

$$f(M) = (p + \delta p) \cdot f(M + P) + (q - \delta p) \cdot f(M - Q). \tag{27}$$

Assume (1) that P and Q are small in comparison to M, (2) that M is so large that the second and higher powers of its reciprocal may be neglected,and (3) that $f(M)$ and its successive derivatives are analytic over the range under consideration. We obtain from the Taylor Series expansion of the members of equation (27) the following differential equation, provided we neglect terms involving third and higher derivatives

$$f''(M) + \frac{\log 2}{M} f'(M) = 0. \tag{28}$$

Equation (28) has the solution

$$f(M) = K_1 \cdot M^{(1-\log 2)} + K_2 \tag{29}$$

where K_1 and K_2 are constants of integration. It is easy to verify from this solution that neglect of the third and higher order derivatives in the expansion of the members of equation (27) is justifiable. Since values of the constants of integration are not necessary for the determination of the physical value of the expectation, we, therefore, may write

$$f(M) = M^{.307} \tag{30}$$

for the transformation function which represents the moral disadvantage from the possibility that one of limited means will exhaust his fortune by repetition of a fair venture many times. This function would seem to represent the reasonable minimum of moral advantage (or disadvantage) that would accrue to a person of limited means when he assumes a risk that would augment (or diminish) his fortune.

7. Concluding remarks. In contrast to the transformation function just determined, which seems to represent a reasonable minimum of moral advantage (or disadvantage), there also seems to be a limit in the other direction. Consider

the physical value of probability of entire loss of a fortune. The equation for determining the physical value, X, of such expectation is

$$f(M - X) = p \cdot f(0) + (1 - p) \cdot f(M), \tag{31}$$

where M is the amount of the fortune and p is the probability of loss. If $f(M)$ is a positive power (not exceeding unity) of M, equation (31) yields a value for X less than M. If $f(M)$ is $\log M$, as in Bernoulli's Hypothesis, or a negative power of M, equation (31) yields X equal to M, that is, no matter how small the probability of loss may be, the physical value of the expectation of loss is equal to the fortune itself. Clearly, this last result is untenable, for it is generally agreed that there is some probability, however small, that one may lose his entire fortune, and no one would value the expectation of such loss as highly as the fortune itself, when the probability of the loss is not in the neighborhood of certainty.

DISCUSSIONS AND NOTES

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The Department of Dicussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

AZIMUTHAL EQUIDISTANT PROJECTION OF THE SPHERE*

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The azimuthal equidistant map projection is a graphic means of showing the great circle distances and great circle bearings of points on the earth from any given point. The given point is the center of the map, and any point is mapped by its great circle distance from the center and the initial great circle course from the center to the point.

In Figure 1, O is the point which is to be the center of the map; the coordinates of any point A are the great circle arc OA, and the angle P_NOA of the spherical triangle P_NOA. Angle P_NOA is the initial great circle course from O to A; it is measured clockwise from 000°, north, through 360°. Angle P_NOA is also called the true bearing of A from O.

The azimuthal equidistant projection has rather obvious uses in showing great circle routes from any point. "The projection has been employed in celestial maps, in problems of crystalloptics by the Carnegie Geophysical Laboratory, by the National Geographic Society for maps of the Arctic Regions, North America,

* In accordance with the regulations of the United States Navy, this article is to be understood as reflecting the writer's personal opinions, and is not to be construed as official or as reflecting the views of the Navy Department or the naval service at large.

Asia, and Africa, by the General Electric Company for a map of the world."*

The method for constructing this projection, given by Deetz and Adams, consists in calculating the coordinates of the intersections of the meridians and parallels. It is the purpose of this note to present an alternate method which it is believed will shorten the computations. The notation used follows that of Dutton's *Navigation and Nautical Astronomy*.

To construct a meridian: Let O, latitude L_0, longitude λ_0, be the center of the projection and P_NAP_SB the great circle formed by two meridians differing 180° in longitude (Fig. 1). Let the difference of longitude from O to A, DLO_A, be less than the difference of longitude from O to B, DLO_B.

Construct the great circle AOB perpendicular to the meridians PA and PB. Denote the great circle distance of A from O by D_A and the initial great circle course from O to A by C_A, *i.e.*, the map coordinates of A are C_A and D_A. In right triangle P_NOA (Fig. 2):

$$\sin D_A = \cos L_0 \sin DLO_A, \tag{1}$$

$$\cot C_A = \sin L_0 \tan DLO_A. \tag{2}$$

The coordinates of B are C_B and D_B:

$$C_B = 180° + C_A,$$
$$D_B = 180° - D_A.$$

For other points on the meridian it is easy to determine the value of the initial great circle course which corresponds to a great circle distance D_X, where $D_A < D_X < D_B$. D_X determines two points X_1 and X_2 such that,

$$C_{X_1} = C_A - \alpha_X,$$
$$C_{X_2} = C_A + \alpha_X.$$

From right triangle X_1OA (see Fig. 2),

$$\cos \alpha_X = \tan D_A \cot D_X. \tag{3}$$

Formulas (1), (2), and (3) afford a means of determining the coordinates for a given meridian by assuming values of D_X between D_A and D_B.

To construct a parallel: It is convenient to assume appropriate values of D and calculate the corresponding values of C. For the parallel of L_1,

$$L_0 < L_1 < 90°, \qquad L_1 - L_0 \leqq D \leqq 180° - (L_0 + L_1),$$

$C = 000°$ for both the maximum and minimum values of D. At two points E and F of L_1, the great circles OE and OF are tangent to L_1 (Fig. 3). In right triangle OP_NE:

* C. H. Deetz, and Adams, O. S. Elements of Map Projections, Special Publication No. 68, U. S. Coast and Geodetic Survey, page 167.

$$\cos D_E = \sin L_0 \csc L_1, \tag{4}$$

$$\sin C_E = \sec L_0 \cos L_1, \tag{5}$$

$$D_F = D_E, \qquad C_F = 360° - C_E.$$

Hence for L_1 between L_0 and $+90°$, C varies from 000° to C_E and from 360° to C_F.

For parallels of latitude between L_0 and $-L_0$, C varies from 000° to 360°. The point diametrically opposite to L_0, λ_0, *i.e.*, $-L_0, \lambda_0 \pm 180°$, is mapped by the periphery of the projection. For parallels of latitude between $-L_1$ and $-90°$, $C=180°$ for both the maximum and minimum values of D, and C is restricted to a certain range on either side of 180°.

If values of D are assumed for a given parallel, then it is sufficient to solve a spherical triangle of sides $90°-L_0$, D, and $90°-L_1$, for angle C opposite $90°-L_1$. In a spherical triangle of sides a, b, and c,

$$\sin^2 C/2 = \sin (s - a) \sin (s - b) \csc a \csc b.$$

Let $2S=L_1+L_0+D$, $c=90°-L_1$, $b=90°-L_0$, and $a=D$, then $s-b=S-L_1$ and $s-a=90°-S$. It is convenient to replace $\sin C/2$ by the haversine function, $\text{hav}\, C=(1-\cos C)/2=\sin^2 C/2$. The above formula may be written,

$$\text{hav}\, C = \sec L_0 \sin (S - L_1) \cos S \csc D. \tag{6}$$

Values of C corresponding to appropriate values of D may be readily computed by formula (6).

Example 1. Center 40° 46′ N., 73° 52′ W.

(a) Meridian of 30° W. 150° E.:

$$D_A = 31°\,39', \quad D_B = 148°\,21', \quad C_A = 057°\,53', \quad C_B = 237°\,53'.$$

(b) Parallel of 60° N.:

$$D_E = D_F = 41°\,04', \quad C_E = 041°\,19', \quad C_F = 318°\,41'.$$

(c) Parallel of 60° S.:

$$D_E = D_F = 138°\,56', \quad C_E = 138°\,41', \quad C_F = 221°\,19'.$$

Example 2. Center 33° 52′ S., 151° 13′ E.

(a) Meridian of 0°−180°:

$$D_A = 23°\,35', \quad D_B = 156°\,25', \quad C_A = 107°\,02', \quad C_B = 287°\,02'.$$

(b) Parallel of 60° N.:

$$D_E = D_F = 130°\,03', \quad C_E = 037°\,02', \quad C_F = 322°\,58'.$$

(c) Parallel of 60° S.:

$$D_E = D_F = 49°\,57', \quad C_E = 142°\,58', \quad C_F = 217°\,02'.$$

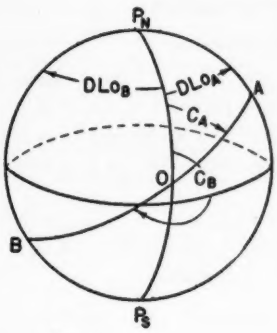


FIG. 1

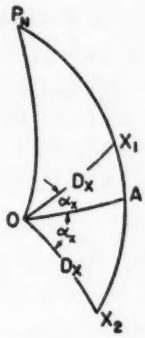


FIG. 2

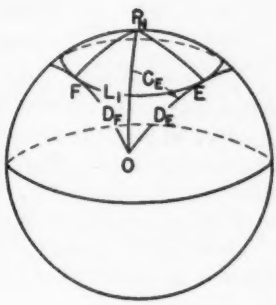
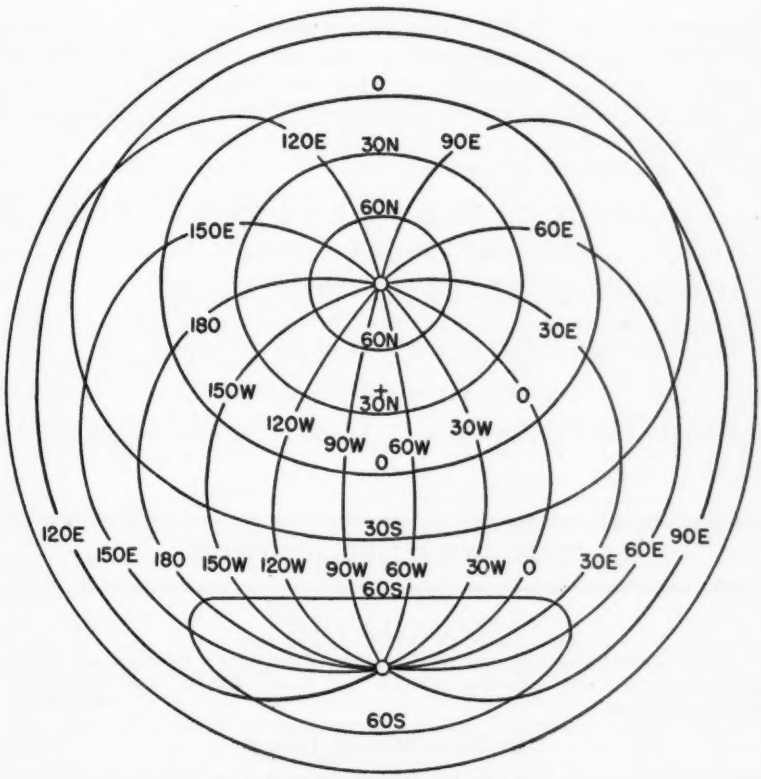


FIG. 3



Azimuthal Equidistant Projection
Center: 40° 46′ N, 73° 52′ W

VARIATIONS OF VECTOR INEQUALITIES

HERBERT RYSER, University of Wisconsin

The vectors $\alpha=(a_1, a_2, \cdots, a_n)$ and $\beta=(b_1, b_2, \cdots, b_n)$ are in an n-dimensional space over the real field. By (α, β), the inner product of α and β, is meant $\sum a_i b_i$, and $\|\alpha\|$, the norm of α, is defined as $(\alpha, \alpha)^{1/2}$. Two inequalities of vector algebra are $|(\alpha, \beta)| \leqq \|\alpha\| \, \|\beta\|$ (the Schwarz inequality) and $\|\alpha+\beta\| \leqq \|\alpha\|+\|\beta\|$ (the triangle inequality).

From α and β, $\lambda=(M_1, M_2, \cdots, M_n)$ and $\mu=(m_1, m_2, \cdots, m_n)$ are formed by choosing $M_i=\max(|a_i|, |b_i|)$ and $m_i=\min(|a_i|, |b_i|)$, for $i=1, 2, \cdots, n$. These λ, μ vectors offer variations of the above inequalities.

$$|(\alpha, \beta)| \leqq \|\lambda\|\|\mu\| \leqq \|\alpha\| \, \|\beta\| \tag{1}$$

$$\|\alpha+\beta\| \leqq \|\lambda\| + \|\mu\| \leqq \|\alpha\| + \|\beta\|. \tag{2}$$

To prove (1), apply Schwarz's inequality to λ and μ, giving $|(\lambda, \mu)| \leqq \|\lambda\| \, \|\mu\|$. Since $|(\alpha, \beta)| \leqq (\lambda, \mu)$, it remains only to establish that $\|\lambda\|\|\mu\| \leqq \|\alpha\| \, \|\beta\|$. From the choice of λ and μ, $\|\alpha\| \leqq \|\lambda\|$, $\|\mu\| \leqq \|\beta\|$, and $\|\beta\| \leqq \|\lambda\|$, $\|\mu\| \leqq \|\alpha\|$. Hence $|\|\alpha\|-\|\beta\|| \leqq \|\lambda\|-\|\mu\|$ and $\|\alpha\|^2-2\|\alpha\| \, \|\beta\|+\|\beta\|^2 \leqq \|\lambda\|^2-2\|\lambda\|\|\mu\|+\|\mu\|^2$. But $\|\alpha\|^2+\|\beta\|^2=\|\lambda\|^2+\|\mu\|^2$. Thus $\|\lambda\| \, \|\mu\| \leqq \|\alpha\| \, \|\beta\|$.

For the derivation of (2) note that

$$\|\alpha+\beta\|^2 = (\alpha, \alpha) + 2(\alpha, \beta) + (\beta, \beta) = \|\alpha\|^2 + 2(\alpha, \beta) + \|\beta\|^2$$
$$\leqq \|\lambda\|^2 + 2(\lambda, \mu) + \|\mu\|^2 = \|\lambda+\mu\|^2,$$

and $\|\lambda+\mu\|^2 \leqq \|\lambda\|^2+2\|\lambda\| \, \|\mu\|+\|\mu\|^2 \leqq \|\alpha\|^2+2\|\alpha\| \, \|\beta\|+\|\beta\|^2$.

The angle θ between α and β is defined as the arc cos $[(\alpha, \beta)/\|\alpha\| \, \|\beta\|]$, $0 \leqq \theta \leqq \pi$. If ϕ denotes the angle between λ and μ, inequality (1) implies

$$0 \leqq \phi \leqq \theta \leqq \pi. \tag{3}$$

Inequalities which are analogues of (1) and (2) hold for infinite series and functions.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

Pi Mu Epsilon, Marquette University

The activities of *Pi Mu Epsilon* during the past year have been affected by the war and the entrance of many active members into the service. Nevertheless, the fraternity continued its monthly meetings and attempted to further

mathematical interest. Five new members were initiated in November: Donald Bennett, Beverly Ullmer, Gary Himler, Eugene Philipp, and Charles Polzin.

The writing of the history of the local chapter and of the national organization was one of the projects undertaken by the fraternity. Papers presented during the year included:

The significance of exponents, by E. A. Halbach

Celestial navigation, by Dr. Ross Bardell, of the University of Wisconsin in Milwaukee. This was the after dinner speech at the annual banquet in May. At this time Donald La Budde, as winner of the Frumveller contest for high school seniors, was presented with a $100 scholarship to Marquette University.

Officers for the coming year are: Director, A. J. Gillan; Vice-Director, Gary Himler; Corresponding Secretary, Lillian Schnell; Recording Secretary, Lois Ebert; Treasurer, Mrs. Madeline Kennedy; Librarian, Beverly Ullmer.

Mathematics Club, Boston University

The Club held monthly meetings at which the following talks were presented:

Plotting of graphs by mechanical means, by Francis Scheid

Statistics, by Robert Haskins of Harvard University

Picturing fractions, by Peter Franck of Harvard University

Analytical triangles, by Emma De le Vin

As an extra activity, some of the members visited the meetings of the Harvard Mathematics Clubs during the winter. The members also visited the Treasure Room of the Boston Public Library and examined rare mathematics texts preserved there.

Officers for the year 1943–44 were: President, Mary Siteman; Vice-President, Emma De le Vin; Secretary, Marion King; Treasurer, Doris Stovald; Faculty Adviser, Professor Lewis Brigham. Officers elect for 1944–45 are: President Harry Kouyoumjian; Vice-President, Marion King; Secretary, June Westgate; Treasurer, Phyllis Paulsen; Faculty Adviser, Professor Ralph Johanson.

Mathematics Club, Immaculate Heart College

The Mathematics Club of Immaculate Heart College, Hollywood, California, organized in January 1944, held monthly meetings from January to June, at which papers were presented by students. Topics discussed were:

Descartes—Analytic geometry, by Zilda Cross

Desargues—Projective geometry, by Sister Margaret Ann

De Moivre's theorem, by Kitty Campion and Jean Gormaly

Mathematics in science and aircraft, by Peggy Haws.

The Club is directed by Dr. Myrtle Collier. The officers were: President, Zilda Cross; Secretary-Treasurer, Sister Margaret Ann.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Marine and Air Navigation. J. Q. Stewart and N. L. Pierce. Boston, Ginn and Co., 1944. 12+472 pages. $4.50.

At the outbreak of the present war, a large number of short texts on navigation began to come from the publishers. The object was to shorten and clarify the standard texts, Bowditch's *American Practical Navigator* and Dutton's *Navigation and Nautical Astronomy*. The former, written in 1802, was completely revised in 1938. Benjamin Dutton, Commander U. S. Navy, wrote his text in 1926. The last chapter in the fifth edition, 1934, contained forty pages entitled "Aerial Navigation."

Some of the recent texts deal only with air navigation; some pay most attention to the recently developed short methods of navigation. Actually the fundamental principles of marine and air navigation are the same.

The authors of this text—*Marine and Air Navigation*—state: "Our experience in the teaching of marine and air navigation for a number of years has convinced us of the need for a comprehensive and understandable textbook—one which clearly explains present-day methods and practice." Their book is certainly comprehensive. It contains 400 pages of text—two columns to the page of $8\frac{1}{2}\times 11$ paper—18 pages of drill problems with answers, and eight star charts. There are 367 photographs and diagrams which are numbered by chapters and 47 tables including excerpts from Bowditch's tables and various Hydrographic Office publications.

The photographs are numerous and include remarkable shots of naval vessels from official sources, the interiors of army training planes showing navigation instruments, and astronomical photographs to illustrate a chapter on Astronomy for Navigators. One has a feeling that the photographs, copies of charts, and drawings are the best part of the book.

The text covers approximately the same subjects as the standard texts by Bowditch and Dutton, with more emphasis on air navigation. The surface navigator would want to omit much on air navigation and the aviator would not need much of the material on marine navigation. It is easy to see why the Air Corps prefers its own manuals.

Many students object to the difficulty of reading Dutton's book because it includes too much material. This text has the same fault, although the authors try to make it more readable by numerous incidents or words of advice to the reader. We wonder if the extra words make the study of navigation more understandable. The many excellent exercises completely worked out should serve the

purpose sufficiently. It is enjoyable to read the history of navigation which is included.

The star charts are excellent—a big improvement over those furnished in the Nautical Almanac. The necessary astronomical terms for celestial navigation are explained with diagrams. Since the seven pages on Astronomy of General Interest are too brief to be of much interest, the student would find a short text on Astronomy more satisfactory.

Two unusual tables are worth mentioning. One is a double page of Tolerances, giving the degree of accuracy which is to be expected in various phases of navigation. The other is a long-term almanac for the sun and stars. This is a table of corrections to the 1940 almanac and is good until 1955.

The publishers have done an unusual job by producing a text on excellent paper, with large, readable type and free from errors—a notable accomplishment in war time. The photographs make it especially attractive. Since each school has its own preferences for methods, texts and references, this book should find a useful place in the teaching of navigation.

C. M. HUFFER

Navigational Handbook with Tables. By T. F. Hickerson. Chapel Hill, N. C. Published by the Author, 1944. 1+77 pages. $1.50.

The principal part of this "Handbook" is a 45-page expansion of the Ageton Table (H. O. 211) of logarithmic cosecants and secants, in which angles are given to each 0.2 minute of arc instead of the usual 0.5 minute. This should make interpolation unnecessary, or at least simpler, in "solving the usual problems of navigation." The number of pages required has been cut in half by a slight reorganization of the column headings. This change necessitates modification of the rules for determination of the quadrant of Z, the new rules being printed in bold-face type on the pages of work forms instead of on each page of the table as in H. O. 211.

The book also contains a condensed table of meridional parts, an extensive table for "Conversion of Time into Arc and Vice Versa," and the usual tables of corrections to observed altitudes.

Probably the most useful part of the book is the discussion of the uses of the tables as found in the first 29 pages. Here the author solves 21 examples in detail, deriving the necessary formulas and in the various cases presenting 12 systematic work forms, some of which are new. In addition, he gives 28 problems, with answers.

Notation in the main is standard except for the use of positive and negative signs instead of "names" (north and south), which should appeal to mathematicians if not to practical navigators.

The typography is clean and well spaced; but the binding is inadequate and the book can not be expected to hold together long under normal usage.

R. C. HUFFER

Elements of Trigonometry. Plane and Spherical with Applications. By L. M. Kells, W. F. Kern, J. R. Bland, and J. B. Orleans. New York and London, McGraw-Hill Book Company, Inc., 1943. 10+363 pages. $1.80.

This volume, which includes both plane and spherical trigonometry, has 194 pages on plane trigonometry, plus 54 pages in two chapters on logarithms and the slide rule, and 85 pages on spherical trigonometry. It is, according to the authors, an attempt to tone down the text by Kells, Kern and Bland to the level of understanding of third and fourth year high school students. Insofar as the attractiveness of treatment, readable language and good illustrations are concerned, they have done well.

The early introduction to the solution of the right triangle and application to problems of surveying, plane sailing, and vectors, helps to impress upon the student the practical importance of the subject. The very large number of exercises and problems, while offering a challenge to the superior student, imposes a real responsibility upon the teacher, for a judicious selection of problems to be deferred, as the authors recommend, for later practice and review.

By placing the discussion of logarithms in a separate chapter near the end of the book the right triangle presents a direct, unified topic. It is probable that the study of this unit would have to be interrupted in order to insure the availability of this tool, whose application comes rather early.

In these days, when more people are becoming familiar with the slide rule, the presence of a chapter on its use, and problems designed for slide rule solution is valuable.

One of the outstanding features of the text is the large number of practical and well illustrated problems in applied fields, notably in navigation.

While, supposedly, high school students will be chiefly interested in the applications to problems in mensuration, yet the authors have included more exercises in identities than are to be found in many college textbooks.

One might take issue with their definitions of principal values for inverse functions of negative quantities, which are at variance with usual practice; *e.g.*, $\csc^{-1}(-a)$ is a negative angle with terminal line in the third quadrant while $\sin^{-1}(-1/a)$ is a negative angle terminating in the fourth quadrant.

The textual material would certainly be adequate for most college courses in trigonometry. The extensive offering of exercises and problems (well over 1000 in 57 sets) calls for careful selection for a one semester course.

The 85 pages devoted to spherical trigonometry would, as the authors note, form a good separate course. If a teacher wished to include any substantial amount of this in the one semester course in trigonometry, a somewhat drastic cutting of the matter in plane trigonometry would be necessary.

The usual development of formulas for spherical trigonometry is beautifully supplemented by discussion of and problems in navigation, as one might readily expect, from the background of the authors.

Only two typographical errors were noted, and those were not serious.

H. P. PETTIT

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General Mathematics in American Colleges. By Kenneth Brown. New York, Bureau of Publications, Teachers College, Columbia University, 1943. 1+167 pages. $2.35.

The development of "general mathematics" courses in American colleges, stemming from the pioneer urgings of John Perry (England), Felix Klein (Germany), and E. H. Moore (The United States), should be of great interest to teachers of mathematics. Especially after a forty-year climb does the present status of the movement warrant study. That the stature of general mathematics is by now large is readily inferred from the list of text-books on the subject imposingly arrayed in the excellent bibliography in the volume under review. That its study has occupied a large number of investigators bent on improving the teaching of mathematics is apparent in the list of magazine articles given in the same bibliography. It is time for a book like the one here reviewed, offering both an historical account of the trend and an evaluation of its achievements.

By general mathematics the author means a "modern non-compartmental" course in freshman mathematics. Such a course may be offered to students interested in "tool" mathematics and further work in the subject, or to students who wish a "cultural" course presenting in itself alone the mathematical knowledge and appreciation which is proper for an intelligent citizen. The author's investigation of such courses (based on questionnaires sent to students and instructors in many colleges) indicates that those organized directly for only one of the above types of student achieve a certain amount of success, but single courses aimed in both directions at once fail to achieve results in either direction. Extracts from student opinion concerning these courses reveal little thought beyond the student-bromidic. Samples of procedure in classes visited by the author disclose teaching methods that are both good and (in one case) appallingly and humorously bad. Such evidence can, of course, be held flimsy enough to vitiate the author's conclusions.

The author is even more thorough in his examination of text-books. And the evidence, by quotation from the printed words (mostly prefaces) of the authors of these books, is more substantial though somewhat theoretical: what an author wants to achieve and what he actually achieves are often different. Books are classified into three groups: (i) Preparatory, (ii) Cultural, (iii) Cultural-preparatory, with obvious reference to the student-types and courses mentioned previously. They are studied as to objectives, content, emphasis on mathematical rigor, type and number of exercises, style, prerequisites assumed, and testings of the material by actual class-room use. Opinions of (anonymous) instructors concerning tests (anonymous) are rather pointlessly given. The author makes no attempt to label a given book as good, bad, or indifferent—wisely, of course, as professional courtesy might not always have stood the strain. But he concludes that books in class (iii) are generally unsatisfactory, a not unreasonable conclusion since the two parts of the double objective are so incompatible; those in class (i), the most stabilized class of all, give much satisfaction; and those in the newer class (ii) seem to show large and steady improvement. These estimates

of texts likewise describe his final evaluation of the success of the types of courses in general mathematics; in addition, he seems to feel that there is a fruitful future for courses and books of the cultural type.

These things are presented in a style which is serious, earnest, and generally precise, though sometimes repetitive and a trifle wearisome. But exposition-by-quotation, so extensively used, is a dangerous game: Examining one recent cultural text-book on the subject of exercises, the author (page 90) condemns with this quotation, "The exercises have been relegated to the appendix so that continuity of reading may not be interrupted." Standing alone, this sounds a death knell. Yet investigation of the appendix reveals that, among historical and critical notes, there are copious lists of exercises nicely cross-referenced with the appropriate parts of the text; these number well over a thousand; they are carefully graded; some demand researches on the part of the student; many contain judicious hints of procedure for the non-technical student for whom they are designed. Moreover, examination of the book itself discloses that the author leans heavily on exposition by means of exercises, so that he cannot place "little emphasis on formal exercises." This reviewer is convinced that the author of this text must writhe at such misrepresentation—and he should know, for reviewer and author are the same person!

GAYLORD MERRIMAN

NEW BOOKS RECEIVED

Mathematics. Second Edition. By J. W. Breneman. New York and London, McGraw-Hill Book Company, Inc., 1944. 12+224 pages. $1.75.

A Guide to Tables of Bessel Functions. By Harry Bateman and R. C. Archibald. (Vol. 1, Number 7 of Mathematical Tables and other Aids to Computation.) Washington, D. C., Division of Physical Sciences, The National Research Council, 1944. 104 pages. $1.75.

Notre Dame Mathematical Lectures. Numbers 3 and 4. University Press, Notre Dame (Lithoprinted), 1944. Number 3: Algebra of Analysis. By Karl Menger. 50 pages. $1.00. Number 4: Alignment Charts. By L. R. Ford; The Teaching of the Calculus of Probability. By A. H. Copeland; On the Theory of Complex Functions. By Emil Artin. 70 pages. $1.25.

Reports of a Mathematical Colloquium. Second Series, Double Issue 5–6. Edited by Karl Menger. University Press, Notre Dame, 1944. 80 pages. $1.00.

Nautical Mathematics and Marine Navigation. By S. A. Walling, J. C. Hill and C. J. Rees. Cambridge University Press; New York, The Macmillan Company, 1944. 9+221 pages. $2.00.

Vital Mathematics. By E. B. Allen, Dis Maly, and S. H. Starkey, Jr., New York, The Macmillan Company, 1944. 7+456+22 (answers) pages. $1.80.

Alignment Charts, Construction and Use. By M. Kraitchik. New York, Van Nostrand Company, 1944. 94 pages. $2.50.

Engineering Mathematics. By H. Sohon. New York, Van Nostrand Company, 1944. 6+278 pages. $3.50.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 661. *Proposed by Howard Eves, Syracuse University*

A plane p is projected from a point L onto a plane p'. Find those points on p for which all angles on p having such a point for vertex are invariant under the projection.

E 662. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A number is represented by a in the scale of α and by b in the scale of β ($\beta<\alpha$). Regarding both a and b as written in the scale of α, we write the difference $b-a=c$. Show how to determine the greatest possible value of a for given values of α, β, c; *e.g.*, when $\alpha=10$, $\beta=7$, and $c=3501$.

E 663. *Proposed by Irving Kaplansky, Columbia University*

If $2^n+1=p^r$, where p is a prime, prove that r is a power of 2 (including the possibility $r=2^0=1$).

E 664. *Proposed by D. H. Browne, Buffalo, N. Y.*

Prove that if $x<1$,

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}\int_1^{\infty}t^n e^{-t}dt=\frac{e^{x-1}}{1-x}.$$

E 665. *Proposed by L. A. Santaló, Rosario, Argentina*

Let C be a closed convex plane curve with continuous radius of curvature R. Let R_M be the greatest value of R. Given $\lambda\geqq R_M$, show that the area F_λ covered by the centers of circles of radius λ which contain C in their interior is given by

$$F_\lambda=F-L\lambda+\pi\lambda^2,$$

where L and F are the length and area of C.

E 637 [1944, 472]. Hint: Use a plane parallel to one face.

SOLUTIONS

A Tetrahedron whose Opposite Dihedral Angles Are Supplementary

E 617 [1944, 231]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

From a given tetrahedron we derive another by taking as vertices the points of contact of the insphere with the faces. Show that the dihedral angles at pairs of opposite edges of the first tetrahedron are supplementary if, and only if, the second tetrahedron is trirectangular.

Solution by L. M. Kelly, U. S. Coast Guard Academy. (The numbers in parentheses refer to Arts. of Altshiller Court, *Modern Pure Solid Geometry*.) Let $A_1A_2A_3A_4$ be the original tetrahedron, $B_1B_2B_3B_4$ the derived tetrahedron, O and R the incenter and inradius of the former (or the circumcenter and circumradius of the latter). We first observe that the dihedral angles at the edges A_iA_j and A_kA_l are supplementary if and only if the plane angles B_iOB_j and B_kOB_l are supplementary. Thus the problem is reduced to showing that a tetrahedron is trirectangular if, and only if, the angles subtended by pairs of opposite edges at the circumcenter are supplementary.

Suppose first that $B_1B_2B_3B_4$ is given to be trirectangular. Then its vertices belong to a rectangular parallelepiped with center O, and it follows at once from congruent triangles that the desired angles are supplementary.

Now suppose that the angles (such as B_1OB_2 and B_3OB_4) are given to be supplementary. Application of the cosine law easily shows that the sum of the squares of any two opposite edges equals $4R^2$. We conclude (185) that the bimedians are equal, whence (by the converse of 210, which is easily seen to be true), the tetrahedron is orthocentric. From the distance formula

$$OG^2 = R^2 - \sum B_iB_j^2/16,$$

where G is the centroid of $B_1B_2B_3B_4$, we deduce $OG = R/2$. Hence the orthocenter H, being the reflection of O in G, must lie on the circumsphere. The polar sphere has center H and radius $\sqrt{HO^2 - R^2} = 0$. But this radius is also equal to $\sqrt{HB_i \cdot HH_i}$, where H_i is the foot of the altitude from B_i. (See 795 and 823.) Thus H must coincide with either a vertex or the foot of an altitude. In either case the tetrahedron is evidently trirectangular.

Parallel Curves

E 630 [1944, 348 and 405]. *Proposed by R. A. Rosenbaum, U.S.N.R.*

For any point P of a given closed convex curve C, let P' be that point on the exterior normal to C at P for which $PP' = k$, a constant. The locus of P' is a curve C', parallel to C. Let s, s' be the respective lengths of C, C', and A, A' the areas within these curves. Show that

$$s' = s + 2\pi k,$$
$$A' = A + sk + \pi k^2.$$

Solution by Howard Eves, Syracuse University. Neglecting differentials of higher order, we easily see (from a figure showing two adjacent normals) that

$$ds' = ds + kd\phi,$$
$$d(A' - A) = \tfrac{1}{2}k(ds + ds'),$$

where $d\phi$ is the angle between the two adjacent normals. Integrating for a complete circuit, these equations become

$$s' = s + 2\pi k,$$
$$A' - A = \tfrac{1}{2}k(s + s') = \tfrac{1}{2}k(2s + 2\pi k) = sk + \pi k^2.$$

Also solved by Chandler Davis, J. T. Webster, and R. H. Wilson, Jr.

A Cube of the form *abab*

E 632 [1944, 405]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

What is the smallest radix which admits a perfect cube of the form *abab*?

Solution by Colin Blyth, Queen's University. If *abab* is a cube in the scale of r, we have $k^3 = ar^3 + br^2 + ar + b = (r^2+1)(ar+b)$. Here r^2+1 is the larger factor, since

$$ar + b \leqq r(r - 1) + r - 1 = r^2 - 1.$$

Therefore r^2+1 must be divisible by the square of some prime. Testing in turn $r = 2, 3, \cdots$, we find 7 to be the smallest value satisfying this necessary condition. If $r = 7$, we have

$$k^3 = 2 \cdot 5^2(7a + b).$$

Thus k must be divisible by both 2 and 5, say $k = 10k_1$. Then

$$7a + b = 20k_1^3.$$

Since $7a+b<49$, we can only have $k_1 = 1$, $a = 2$, $b = 6$. Thus the radix 7 admits the cube $2626 = 13^3$ (denary $1000 = 10^3$), and is the smallest radix which admits a cube of the form *abab*.

Also solved by D. H. Browne, W. E. Buker, N. J. Fine, Daniel Finkel, Irving Kaplansky, J. B. Kelly, Walter Penney, E. D. Schell, E. P. Starke, F. E. Wood, and the proposer.

It was pointed out that the next possible radix is 38 (with $a = 11$, $b = 7$).

The *n*th Involute of a Circle

E 635 [1944, 405]. *Proposed by R. A. Rosenbaum, U.S.N.R.*

Derive parametric equations for the involute of the involute . . . (n times) of a circle (with the same starting point for each process of unwinding).

Solution by J. B. Kelly, Langley Field, Va. Taking the radius of the circle to be 1, and its center at the origin, we shall prove by induction that the nth involute has the parametric equations $x = x_n(\theta)$, $y = y_n(\theta)$, where

$$x_n(\theta) = 1 + \int_0^\theta \frac{\theta^n}{n!} \cos\left(\theta - \frac{(n-1)\pi}{2}\right) d\theta,$$

$$y_n(\theta) = \int_0^\theta \frac{\theta^n}{n!} \sin\left(\theta - \frac{(n-1)\pi}{2}\right) d\theta.$$

Since $x_0(\theta) = \cos\theta$ and $y_0(\theta) = \sin\theta$, this is true when $n = 0$. So we shall assume the result with $n = m$, and deduce the case when $n = m+1$.

The involute of the curve $x = u(\theta)$, $y = v(\theta)$ has the parametric equations

$$x = u(\theta) - s(\theta)\cos\phi(\theta), \qquad y = v(\theta) - s(\theta)\sin\phi(\theta),$$

where $\phi(\theta)$ is the angle of slope of the curve, and $s(\theta)$ is the length measured from the point where the unwinding commences, in our case (1, 0). Setting $u(\theta) = x_m(\theta)$, $v(\theta) = y_m(\theta)$, we have

$$s(\theta) = \int_0^\theta \sqrt{\left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2}\, d\theta = \int_0^\theta \frac{\theta^m}{m!}\, d\theta = \frac{\theta^{m+1}}{(m+1)!},$$

$$\phi(\theta) = \arctan\left(\frac{dv}{d\theta} \Big/ \frac{du}{d\theta}\right) = \theta - \frac{(m-1)\pi}{2}.$$

Hence the involute of the curve $x = x_m(\theta)$, $y = y_m(\theta)$ is given by

$$x = 1 + \int_0^\theta \frac{\theta^m}{m!} \cos\left(\theta - \frac{(m-1)\pi}{2}\right) d\theta - \frac{\theta^{m+1}}{(m+1)!} \cos\left(\theta - \frac{(m-1)\pi}{2}\right)$$

$$= 1 + \int_0^\theta \frac{\theta^{m+1}}{(m+1)!} \cos\left(\theta - \frac{m\pi}{2}\right) d\theta = x_{m+1}(\theta),$$

$$y = \int_0^\theta \frac{\theta^m}{m!} \sin\left(\theta - \frac{(m-1)\pi}{2}\right) d\theta - \frac{\theta^{m+1}}{(m+1)!} \sin\left(\theta - \frac{(m-1)\pi}{2}\right)$$

$$= \int_0^\theta \frac{\theta^{m+1}}{(m+1)!} \sin\left(\theta - \frac{m\pi}{2}\right) d\theta = y_{m+1}(\theta).$$

This completes the induction.

Also solved by the proposer, in the form

$$x = A\cos\theta + B\sin\theta, \qquad y = A\sin\theta - B\cos\theta,$$

$$A = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + (-1)^{p/2}\frac{\theta^p}{p!},$$

$$B = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots + (-1)^{(q+1)/2}\frac{\theta^q}{q!},$$

$$p = 2\left[\frac{n}{2}\right], \qquad q = 2\left[\frac{n+1}{2}\right] - 1.$$

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4151. *Proposed by B. M. Stewart, Michigan State College*

Let O be a point at which a given curve has a second derivative; let the tangent and normal at O serve respectively as x, y axes, the equation of the curve becoming $y=f(x)$; and let P; (x, y) be a point on the curve, say with positive x. Denote the arc length OP by s and locate the point $S:(s, 0)$. The line SP intersects the y-axis in the point $B:(0, b)$. If R indicates $[1+(y')^2]^{3/2}/y''$, show that the limiting position of the point B as P approaches O is such that $\lim b=3 \lim R$.

This is a generalization of a problem in the calculus of Granville, Smith, and Longley, 1934, p. 177, ex. 20.

4152. *Proposed by W. J. Taylor, Washington, D. C.*

Prove the following trigonometric expansion for the binomial coefficient

$$\frac{N!}{\left(\frac{N+x}{2}\right)!\left(\frac{N-x}{2}\right)!}=\frac{2^N}{N}\sum_{m=1}^{N}\left(\cos\frac{m\pi}{N}\right)^N\cos\frac{m\pi x}{N}, \qquad -N<x<N.$$

4153. *Proposed by V. Thébault, Tennie, Sarthe, France*

A sphere (S), radius r, rolls on the plane of the face BCD of the tetrahedron $ABCD$ so that its center S lies on a fixed sphere concentric with (O) the circumsphere of $ABCD$. Show that: (1) The tetrahedron $BCDS$ is inscribed in a fixed sphere (S'). (2) The sphere tangent interiorly to (S) and passing through the points B, C, D envelopes a fixed sphere concentric with (O). If OS' is prolonged by $S'A'=r/2$, the sphere with center A' and passing through B, C, D is orthogonal to (S).

SOLUTIONS

A Nearly Periodic Product

4100 [1943, 569]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $N=1\,2\,3\cdots n$ be a number in the system of base $n+1$ where N is written with the consecutive increasing digits omitting 0. The product $P=N\cdot L$ is formed where $L=\alpha\beta$ is a number of two digits such that $\gamma=\alpha+\beta$ is a number less

than n and such that γ and n have δ as the greatest common divisor. Show that

$$P = N \cdot L = ab \cdots pqab \cdots pq \cdots ab \cdots p(n-\gamma)q,$$

the $\delta-1$ periods $ab \cdots pq$ being formed by n/δ distinct digits and the number of $(n/\delta)+1$ digits on the right contains the digit $(n-\gamma)$ between the digits p and q.

Dedicated to E. P. Starke.

Solution by E. P. Starke, Rutgers University. This problem is similar to 3851 [1940, 56] and may be proved by a slight modification of the earlier method.

(1) We note first the $(n+1)$-digit number

$$R = n \cdot N = (n+1) \cdot N - N = 111 \cdots 11101.$$

Then the proposed number $P = N \cdot L = R \cdot L/n$. We need also the $(n+2)$-digit number

$$M = R \cdot L = \alpha\gamma\gamma\gamma \cdots \gamma\beta\alpha\beta.$$

The problem is to discuss possible quotients P of the numbers M divided by n. It seems slightly simpler to treat instead the $(n+2)$-digit number

$$K = M + n \cdot L = M + \alpha\beta 0 - \alpha\beta = \alpha\gamma\gamma\gamma \cdots \gamma\gamma\beta 0,$$

of which all digits are γ except the first and the last two.

(2) Any number is congruent, mod n, to the sum of its digits. (Thus M and K are multiples of n.)

(3) In carrying out the division of K by n, the ith and jth remainders, r_i and r_j, $i<j<n$, are alike if and only if the $(j-i+1)$-digit number $r_i\gamma\gamma\gamma \cdots \gamma(\gamma - r_i)$* is divisible by n. By (2) it is congruent, mod n, to $\gamma(j-i)$ which is divisible by n/δ. Hence the number K/n consists of δ periods of n/δ digits each and a final zero.

(4) For the divisions of the 2-digit numbers $\theta\gamma$, $\theta = 0, 1, 2, \cdots, (n-1)$ by n, different remainders imply different quotients. Thus the digits within any one period of K/n are distinct. Further, the penultimate digit of K/n must be $q = (n+1-\beta)$, since the corresponding digit of K is β.

(5) Finally $P = (K/n) - L$ has the same digits as K/n except that the last two digits are $(n-\alpha-\beta)q$ instead of $q0$. This completes the proof.

Editorial Note. The proposer remarked that this completes his theorem in *Mathesis*, 1937, pp. 7–10 where γ and n are relatively prime. He gave the example $n+1=10$, $L=24$, $\delta=3$, $n/\delta=3$

$$P = N \cdot L = 123456789 \times 24 = 2962962936.$$

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* If $r_i > \gamma$, the last two digits are $(\gamma-1)(\gamma+n+1-r_i)$.

Number Theory Determinants

4101 [1943, 637]. *Proposed by R. C. Buck, Harvard University*
Show that

$$D(n) = \begin{vmatrix} (1,1)^\lambda, & (1,2)^\lambda, & \cdots, & (1,n)^\lambda \\ (2,1)^\lambda, & (2,2)^\lambda, & \cdots, & (2,n)^\lambda \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (n,1)^\lambda, & (n,2)^\lambda, & \cdots, & (n,n)^\lambda \end{vmatrix}$$

$$= (n!)^\lambda \left(1 - \frac{1}{2^\lambda}\right)^{[n/2]} \left(1 - \frac{1}{3^\lambda}\right)^{[n/3]} \left(1 - \frac{1}{5^\lambda}\right)^{[n/3]} \cdots$$

where (i, j) means the greatest common divisor of the integers i, j.

Solution by H. S. Zuckerman, University of Washington. We can generalize the problem by considering

$$D_n = |f((i,j))|, \tag{1}$$

where $f(x)$ is defined for all positive integral values of x. To evaluate $D(n)$ we define $\psi(k)$ by the equations

$$f(l) = \sum_{k|l} \psi(k), \tag{2}$$

and we define a_{kl} to be 1 if l divides k, zero otherwise. We then have

$$\sum_l a_{rl} a_{sl} \psi(l) = \sum_{l|(r,s)} \psi(l) = f((r,s))$$

and we can write $D(n) = |a_{rl}|\,|a_{sl}\psi(l)|$ where the determinants on the right are of order n. Since $a_{rl} = 0$ if $r < l$ and $a_{ll} = 1$ we have $|a_{rl}| = 1$ and $|a_{sl}\psi(l)| = \prod_{l=1}^{n} \psi(l)$ and hence $D(n) = \prod_{l=1}^{n} \psi(l)$.

Now inverting (2) by the Möbius inversion formula we have

$$\psi(l) = \sum_{k|l} \mu(k) f\left(\frac{l}{k}\right),$$

and hence the formula

$$D(n) = \prod_{l=1}^{n} \left\{ \sum_{k|l} \mu(k) f\left(\frac{l}{k}\right) \right\}. \tag{3}$$

For the case $f(x) = x^\lambda$ we have

$$\psi(l) = \sum_{k|l} \mu(k) \left(\frac{l}{k}\right)^\lambda = l^\lambda \sum_{k|l} \frac{\mu(k)}{k^\lambda} = l^\lambda \prod_{p|l} \left(1 - \frac{1}{p^\lambda}\right),$$

and hence

$$D(n) = \prod_{l=1}^{n} l^{\lambda} \prod_{p|l}\left(1-\frac{1}{p^{\lambda}}\right) = (n!)^{\lambda} \prod_{p\leq n} \prod_{1\leq l\leq n,\, p|l} \left(1-\frac{1}{p^{\lambda}}\right)$$
$$= (n!)^{\lambda} \prod_{p}\left(1-\frac{1}{p^{\lambda}}\right)^{[n/p]}$$

which is the required result.

Formula (3) can be used to obtain interesting evaluations of (1) in other cases. For example if $f(x)=\delta(x)$, the sum of the divisors of x, it is easily seen that we have

$$\sum_{k|l} \mu(k)\delta\left(\frac{l}{k}\right) = l,$$

and hence the evaluation $D(n)=n!$

It is also possible to invert the process and determine $f(x)$ to obtain a desired $D(n)$. From (3) we have

$$\sum_{k|l} \mu(k) f\left(\frac{l}{k}\right) = \frac{D(n)}{D(n-1)}.$$

Inverting this by the Möbius formula we obtain

$$f(n) = \sum_{k|n} \frac{D(k)}{D(k-1)}.$$

As an example, if we wish to obtain $D(n)=a^n$ we take

$$f(n) = \sum_{k|n} a = a\tau(n)$$

where $\tau(n)$ is the number of divisors of n.

Solved also by C. D. Olds, Robert Steinberg, and the proposer.

Editorial Note. The proposer begins his solution with the definition of $J_\lambda(n)$, using the Möbius inversion, and the function $\delta(i/k)$ equivalent to the a_{ik} above. He remarks that Dickson's History of the Theory of Numbers, vol. 1, p. 123 gives variations, generalizations, and indications of proof of related theorems. Olds also made a similar remark and called attention to the great number of variations cited. The function $J_\lambda(n)$ occurs in the solutions of 4002 [1942, 621] and 4010 [1942, 696].

Digits in a Special Product

4103 [1943, 638]. *Proposed Victor Thébault, Tennie, Sarthe, France*

In the system of base $n+1$ the product $P=N\cdot L$ is formed where the number N of $n-1$ digits in descending order is $n(n-2)(n-3)\cdots 321$, and L is less than n and prime to n. If we have $L<n/2$, then the product P has n distinct digits chosen in suitable order from $0, 1, 2, \cdots, n$, and the missing digit is $n-L$.

If $L>n/2$, the digit $n-1$ appears twice in P, and the missing digits are n and $(n-1-L)$, the remaining digits being distinct.

Dedicated to E. P. Starke.

Solution by E. P. Starke, Rutgers University. We may put $N=R-M$, where R is the n-digit number $11000\cdots0$ and M is the $(n-1)$-digit number $123\cdots(n-2)n$. Note that $n\cdot M$ is the n-digit number $111\cdots1$, so that

$$P = L\cdot N = L\cdot R - L\cdot M$$

is the n-digit number $L\cdot R=LL000\cdots0$ minus the $(n-1)$-digit number $L\cdot M=LLL\cdots L/n$.

A discussion of $L\cdot M$ analogous to the solution of 3851 [1940, 56] follows.

(1) Any number is congruent, mod n, to the sum of its digits.

(2) If $n\cdot L\cdot M=LLL\cdots L$ is divided by n, no two of the partial divisions can produce the same remainder. For if the ith and jth remainders, r_i and $r_j(i<j<n)$ could be equal, the $(j-i+1)$-digit number $r_iLL\cdots L(L-r_j)$* would be divisible by n whereas by (1) it is congruent mod n to $L(j-i)$ with $j-i<n$ and L relatively prime to n. Similarly no remainder r_i can be zero since the number $LL\cdots L$ of less than n digits is not divisible by n.

(3) For the divisions of the two-digit numbers $\theta L(\theta=0, 1, 2, \cdots, n-1)$ by n, different remainders imply different quotients. Thus the $n-1$ digits of $L\cdot M$ are all distinct. The quotient 0 does not occur. The last quotient is $(n-L+1)$. The first partial division, $(n+1)L+L=nL+2L$ by n, indicates that the first digit of $L\cdot M$ is (a) L when $2L<n$ and (b) $(L+1)$ when $2L>n$.

(4) Besides 0, let d be the digit missing from $L\cdot M$. Then $L\cdot M+d\equiv\sum_{k=1}^{n}k$, but $M+(n-1)\equiv\sum_{k=1}^{n}k$, whence $(L-1)\sum k+d+L\equiv0$. From $\sum k=n(n+1)/2$ we see that $\sum k\equiv0$ if n is odd; while for n even, L (prime to n) is odd and $L-1$ is even. Thus always $(L-1)\sum k\equiv0$ and then $d+L\equiv0$ and $d=n-L$. To summarize: $L\cdot M$ has the form

(a) $Lg_2g_3\cdots g_{n-2}(n-L+1)$, (b) $(L+1)g_2g_3\cdots g_{n-2}(n-L+1)$,

according as (a) $L<n/2$ or (b) $L>n/2$, where g_i, $i=2, 3, \cdots, n-2$, are the remaining digits. The digits are distinct and 0 and $n-L$ are missing.

(5) To subtract $L\cdot M$ from $L\cdot R$ we first change the form of $L\cdot R$ by "borrowing" and have in case (a),

$$\begin{array}{lll} L\cdot R = (L-1)(L+n) & & n\,n\cdots n(n+1) \\ L\cdot M = \quad L & & g_2g_3\cdots g_{n-2}(n-L+1) \\ \hline P = (L-1) & n(n-g_2)(n-g_3)\cdots(n-g_{n-2}) & L \end{array}.$$

Since 0, $n-L$, L, $n-L+1$ are not included among the g_i, we see that n, L, $n-L$, $L-1$ are the digits not included among the $(n-g_i)$, but $L-1$, n, and L occur at the ends. Thus the digits of P are all distinct and only $n-L$ is missing.

* If $r_i>L$, the last two digits are $(L-1)(L+n+1-r_i)$.

Case (b) is similar:

$$\begin{array}{lllllll} L\cdot R = (L-1)(L+n) & n & n & \cdots & n & (n+1) \\ L\cdot M = \quad (L+1) & g_2 & g_3 & \cdots & g_{n-2}(n-L+1) & \\ \hline P = (L-1)(n-1)(n-g_2)(n-g_3)\cdots(n-g_{n-2}) & L & . & & & \end{array}$$

Since 0, $n-L$, $L+1$, $n-L+1$ are not among the g_i, then n, L, $n-L-1$, $L-1$ are missing from the $(n-g_i)$. Including $L-1$, $n-1$ and L, we see that in P the digit $n-1$ will occur twice and that n and $n-L-1$ will be missing.

Editorial Note. The proposer gave for $n+1=10$

$$P = 97654321 \times 4 = 390617284,$$

$$P = 97654321 \times 7 = 683580247.$$

The Möbius Function

4104 [1944, 49]. *Revised. Proposed by E. T. Bell, California Institute of Technology*

Two symmetric functions, $M(x_1, \cdots, x_n)$, $S(x_1, \cdots, x_n)$, of n non-negative integers $x_1, \cdots, x_n$ are defined as follows

$$M(x_1, \cdots, x_n) \equiv M'(x_1) \cdots M'(x_n),$$

in which $M'(x)=1, -1, 0$, according as $x=0$, $x=1$, $x>1$; if $S_j(x_1, \cdots, x_n)$ is the jth elementary symmetric function of $x_1, \cdots, x_n$,

$$S(x_1, \cdots, x_n) \equiv 1 + \sum_{j=1}^{n} jS_j(x_1, \cdots, x_n).$$

Prove that $\sum M(x_1-b_1, \cdots, x_n-b_n)S(b_1, \cdots, b_n)$ = the number of integers >0 in the set $x_1, x_2, \cdots, x_n$, and $=1$, if $x_1=x_2=\cdots=x_n=0$, where the summation refers to all integers b_i such that

$$0 \leqq b_i \leqq x_i, \qquad i = 1, 2, \cdots, n.$$

Solution by N. J. Fine, Indianapolis, Ind. Let $p_1, \cdots, p_n$ be any set of distinct primes, and let $N=p_1^{x_1}\cdots p_n^{x_n}$. By definition, $M(x_1, \cdots, x_n)=\mu(N)$. For any divisor m of N, define $f(m)$ as the number of distinct prime factors of m, unless $m=1$; let $f(1)=1$. Further, if d is a divisor of N, $d=p_1^{b_1}\cdots p_n^{b_n}$, define $g(d)=\sum_{m|d} f(m)$. In this sum, group all those m with exactly j distinct factors; the number of such m is clearly $S_j(b_1, \cdots, b_n)$. Adding 1 for $m=1$, we have

$$g(d) = 1 + \sum_{j=1}^{n} jS_j(b_1, \cdots, b_n) = S(b_1, \cdots, b_n).$$

Now, by the Möbius inversion formula,

$$f(N) = \sum_{d|N} \mu(N/d)g(d) = \sum M(x_1 - b_1, \cdots, x_n - b_n)S(b_1, \cdots, b_n).$$

Therefore the required sum is equal to the number of *positive* integers in the set $x_1, \cdots, x_n$, or unity if all are zero.

Solved also by the proposer.

Tetrahedron and Monge Point

4106 [1944, 49]. *Proposed by N. A. Court, University of Oklahoma*

If the Monge point of a tetrahedron lies on the circumsphere, show that (a) The line joining the circumcenter to the centroid of a face is equal to half the corresponding median of the tetrahedron; (b) Each median subtends a right angle at the Monge point. Conversely.

Solution by Howard Eves, Syracuse University. Let the tetrahedron be $ABCD$ with circumcenter O, centroid G, and Monge point M. Let G_a be the centroid of face BCD. Then A, O, G, M, G_a are coplanar, $OM=OA$, $OG=GM$, $AG=3GG_a$. Let K be the centroid of triangle AOM.

(a) OG_aMK is a parallelogram because OM and KG_a bisect each other. Therefore $OG_a=KM=KA=\frac{1}{2}AG_a$.

(b) G_aM is parallel to OK, and therefore perpendicular to AM.

Conversely, if G_aM is perpendicular to AM, then OK is perpendicular to AM and triangle AOM is isosceles, with $OM=OA$. Thus M is on the circumsphere.

We have actually established a stronger converse than is implied by the wording of the problem. We have shown that if *one* median of a tetrahedron subtends a right angle at the Monge point then all the medians do likewise, and the Monge point lies on the circumsphere.

Solved also by L. M. Kelly, P. D. Thomas and the proposer.

Spheres and Powers

4109 [1944, 96]. *Proposed by N. A. Court, University of Oklahoma*

Prove that the sum of the n^2 powers of n given points with respect to the n spheres having for diameters the n segments joining the given points to a variable point in space is constant.

Solution by Louis Brand, University of Cincinnati. LEMMA: *The power of a point P with respect to a sphere having AB as diameter is $\overrightarrow{PA}\cdot\overrightarrow{PB}$.* For if C is the center of the sphere, $\overrightarrow{PA}\cdot\overrightarrow{PB}$ can be written

$$(\overrightarrow{PC}+\overrightarrow{CA})\cdot(\overrightarrow{PC}+\overrightarrow{CB}) = (\overrightarrow{PC}+\overrightarrow{CA})\cdot(\overrightarrow{PC}-\overrightarrow{CA}) = (PC)^2-(CA)^2.$$

Let $P_1, P_2, \cdots, P_n$ be the n given points, P the variable point; and let p_{ij} denote the power of P_i with respect to the sphere on P_jP as diameter. Then, from the lemma above,

$$p_{ij}+p_{ji} = \overrightarrow{P_iP_j}\cdot\overrightarrow{P_iP}+\overrightarrow{P_jP_i}\cdot\overrightarrow{P_jP} = \overrightarrow{P_iP_j}\cdot\overrightarrow{P_iP_j};$$

hence the sum of the n^2 powers p_{ij} is equal to the sum of the squares of the $\frac{1}{2}n(n-1)$ distances between the points P_i.

This sum can be expressed in another form. For if $\mathbf{r}_i$ and $\mathbf{r}^*$ denote position vectors of the points P_i and of their mean center P^*, and $\mathbf{r}=\overrightarrow{OP}$,

$$\begin{aligned}\sum_{ij} p_{ij} &= \sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbf{r}_j-\mathbf{r}_i)\cdot(\mathbf{r}-\mathbf{r}_i)\\ &= n\sum_{i=1}^{n}(\mathbf{r}^*-\mathbf{r}_i)\cdot(\mathbf{r}-\mathbf{r}_i)\\ &= n\left(n\mathbf{r}^*\cdot\mathbf{r}-n\mathbf{r}^*\cdot\mathbf{r}^*-n\mathbf{r}^*\cdot\mathbf{r}+\sum_{1}^{n}\mathbf{r}_i\cdot\mathbf{r}_i\right)\\ &= n\sum_{1}^{n}\mathbf{r}_i^2-n^2(\mathbf{r}^*)^2.\end{aligned}$$

If the origin is chosen at P^*,

$$\sum_{ij} p_{ij} = n\sum_{1}^{n}\mathbf{r}_i^2.$$

Solved also by H. Eves, Irving Kaplansky, L. M. Kelly, C. E. Springer, Robert Steinberg, P. D. Thomas, J. A. Zilber, and the proposer.

Eves remarked that the theorem and proof are valid for any euclidean space.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. C. L. Clark of the University of Virginia has been appointed to an assistant professorship at Oregon State College.

Assistant Professor W. M. Davis of Cornell College, Mt. Vernon, Iowa, has been promoted to a professorship.

Adjunct Professor W. S. Krogdahl of the University of South Carolina has been promoted to an associate professorship.

Professor C. J. Rees of the University of Delaware has been granted leave of absence for civilian service with the Fourteenth Air Force in China. In his absence Assistant Professor B. C. Webber is acting chairman of the mathematics department.

Professor H. A. DoBell of the New York State College for Teachers at Albany died December 8, 1944.

Professor W. P. Ott of the University of Alabama died December 25, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

NEW SELECTIVE SERVICE REGULATIONS

Upon December 12, 1944, the Selective Service System released a statement pertaining to a stricter application of Local Board Memorandum No. 115 on occupational classification. The first part of this statement is reproduced below. One sentence has been emphasized by the use of italics because of the great amount of discussion which it has caused.

"Since May 1944,* the provisions of Local Board Memorandum No. 115 have been strictly applied for registrants ages 18 through 25 and liberally applied for registrants ages 26 through 37 in order to fill calls substantially from younger men capable of the highest degree of efficiency under combat conditions. Estimates of available men indicate that the armed forces calls after February 1, 1945 cannot be filled substantially from men ages 18 through 25. The larger number of American Divisons now actually engaged in combat, and the continuing pressure against the enemy has increased the requirements for physically fit soldiers and sailors. For these reasons, it will be necessary to induct increasing numbers of men from the older age group.

"The larger number of American Divisions now actually engaged in combat, and the continuing pressure against the enemy has also greatly expanded the requirements for military supplies of all sorts. This has resulted in a sharp upwards revision of many war production schedules, requiring a substantial number of additional workers in direct war production activities.

"Because of these requirements, it is imperative that local boards fill calls for the armed forces, and that they be filled by the reclassification, as it becomes necessary, of men in the older age groups who do not meet a stricter application of the provisions of Local Board Memorandum No. 115 in the light of the immediate urgencies for men in the armed forces and the civilian war effort.

"In applying the tests for occupational deferment for registrants ages 26 through 37, greater consideration will be given to registrants now engaged, or who become engaged, in war production or in support of the war effort, than to those engaged in activities not supporting the immediate prosecution of the war. *Registrants of lesser skills may be more important in war production or activities directly supporting the war effort than those of greater skills in other activities not in direct support of the war effort.*

* Note the article, "Deferment of Mathematicians Ages 26 through 37," in this MONTHLY for August–September, 1944.

"Under the stricter interpretation of Local Board Memorandum No. 115 for a registrant age 26 through 37, local boards will consider the continuance of occupational deferment on the basis of general information available, specific information in the registrant's file, representations by other Federal Government agencies, and the local board's own knowledge of the relative importance of civilian activities and the labor supply conditions existing in the area in which the registrant is working."

NOTES ON THE NAVY V-12 PROGRAM

The following quotation is from Navy V-12 Bulletin, No. 280, issued December 23, 1944.

"Screening procedures used in preceding terms for assigning subject trainees to upper-level curricula will not be repeated during the current term. A new plan for the assignment of these students to appropriate upper-level courses will be announced at an early date. The Commanding Officers of Navy V-12 Units are directed to inform trainees and academic authorities that the new plan will provide for the continuation in the V-12 Program of all trainees now on board, under the same general conditions as heretofore. No trainee now under instruction will be separated from the Program for any reason other than academic failure, lack of officer-like qualities, breach of discipline, or lack of physical qualifications. It is anticipated that the new plan, when announced, will give subject students greater rather than less freedom in the selection of academic majors and the election of courses of study beyond the second term."—L. E. Denfeld, The Assistant Chief of Naval Personnel.

Recent communications from the Navy Department indicate that the Navy intends to continue the V-12 Program after July 1, 1945. Institutional participation in the Program after that date will undoubtedly depend on several factors, including the result of Congressional debate upon a proposal to extend the Naval ROTC Program. The following excerpts from Navy V-12 Bulletin, No. 278, tell of some plans which have been developed for the V-12 term, starting July 1, 1945.

"This Bulletin outlines procedures to be followed in the selection and transfer to the Navy V-12 Program of applicants to be assigned to college training on July 1, 1945. It is anticipated that those men selected will be ultimately transferred to the Naval Reserve Officer Training Corps. . . . As it is anticipated that the entire available quota for this sixth increment of the Navy V-12 Program will consist of approximately 2000 enlisted men, it is imperative that only outstanding applicants who are fully qualified in all respects be recommended for such training by Commanding Officers. The primary purpose of the Program is to give prospective Naval Officers appropriate training at the college level in those fields of study most useful to the Navy in accordance with its needs. Length of training offered to each successful applicant will depend on (a) his

previous college education, if any, (b) the type of specialized courses for which he is qualified and to which in the discretion of the Navy he is assigned, and (c) his continued demonstration of satisfactory scholarship and officer-like qualities. Applicants will be permitted to express a preference for the type of duty (Deck, Engineering and Supply Corps only) toward which they wish their training to point.... Commanding Officers are directed to consider for transfer to the Navy V-12 Program only those applicants who (a) apply voluntarily for such training, (b) understand fully that an extended period of college academic training is involved, (c) meet all prescribed requirements, and (d) are considered to be definitely outstanding for training as Officer Candidates.

"Each applicant for this program must meet the following (academic) requirements: (1) Be a high school graduate or have been in attendance at or accepted for admission by an accredited college or university. High school or college transcript must show successful completion of courses in elementary algebra and plane geometry; additional courses in mathematics and physics are desirable. (2) Have passed the O'Rourke General Classification Test (given prior to 15 June 1943) with a score of 88 or above; or the new General Classification Test, Forms 1, 2, 3, or 1s (given subsequent to June 15, 1943) with a score of 60 or above. (In the case of an applicant for whom no General Classification Test score is available, it is recommended that an appropriate written and/or oral test be given to determine whether he is properly qualified to pursue successfully a college curriculum generally considered to be more exacting and more difficult than a normal course at a liberal arts college, and to obviate the selection of men who would subsequently have to be returned to general duty for failure to meet minimum educational requirements.)... Enlisted men who have had more than two years of college cannot be recommended for this program; men who have successfully completed two or more years of college work are eligible for the Reserve Midshipman Program, Class V-7."

ACTIVITIES OF THE COMMITTEE ON EXAMINATIONS

W. T. REID, Northwestern University

In June, 1944, Professor M. H. Stone, Chairman of the War Policy Committee of the A. M. S. and the M. A. A., appointed a Subcommittee on Examinations to advise the Examinations Staff of the United States Armed Forces Institute in connection with examinations in mathematics at the college level to be sponsored by the U. S. A. F. I. This Subcommittee was instituted at the request of Professor R. W. Tyler, Director of the Examinations Staff for the U. S. A. F. I., following initial contact between Professor Tyler and the War Training Programs Subcommittee of the War Policy Committee. The following individuals were appointed to this Subcommittee: Professors Ralph Beatley (Harvard); L. L. Dines (Carnegie Institute of Technology); W. L. Hart (Minnesota); C. C. MacDuffee (Wisconsin); W. T. Reid (Northwestern; Chicago, at time of appointment), Chairman.

At a meeting of the Subcommittee held early in July 1944, it was decided to concentrate on perfecting the U. S. A. F. I. subject examinations for College Algebra, Plane Trigonometry, Analytic Geometry, and Calculus. The Subcommittee devoted its efforts to this task from that time until November 30, 1944, when the examinations construction project of the U. S. A. F. I. was terminated by the War Department.

The purpose of these subject examinations is to provide an instrument for evaluating the work of men in courses taken under the auspices of the U. S. A. F. I. A large number of men have received some mathematical training while in the Armed Forces, and these examinations are planned to help meet the need for some means of providing fair credit and proper placement for these men when they return to collegiate institutions. The examinations are of the multiple choice type; the revised forms prepared by this Subcommittee have been planned for examination periods three hours in length. The selection of material covered by these examinations was based on results of an extensive sampling of the opinions of college departments of mathematics in regard to the content of courses in the considered subjects. These opinions were secured, at the suggestion of Professor Tyler, by circulating in June, 1944, a questionnaire prepared by Professor Reid, with the aid of Professor Hart.

Each of these examinations has been prepared in two forms. One of these forms, referred to as Form A, is the confidential form administered only under safeguards provided by the U. S. A. F. I. The War Department has made arrangements for the Form A of the examinations to be given under the following three conditions: firstly, by the Armed Forces Institute for men and women in service; secondly, by the Veterans Administration for those who seek vocational and educational guidance from one of the Veterans Advisement Centers; thirdly, by the Veterans Testing Service which is being set up on the campus of the University of Chicago by the American Council on Education. Copies of the other form, Form B, of these examinations may be purchased from the American Council on Education by faculty members of educational institutions; orders for these tests should be addressed to Cooperative Test Service, 15 Amsterdam Avenue, New York, N.Y. Some institutions may desire to use results on Form B of the examinations in granting credit for work done with the U. S. A. F. I. If an institution desires to use Form A of one of these examinations in accrediting the work of a given individual, however, application should be made to the Veterans Testing Service, The University of Chicago, 6010 Dorchester Avenue, Chicago 37, Ill. In response to such an application a copy of the examination will be issued to a duly appointed representative of the institution for administration, and subsequent return of examination and answer sheet to Chicago for scoring and recording.

In view of the late date of preparation, the forms of these subject examinations prepared by this Subcommittee have not been standardized by the Armed Forces Institute. Both forms A and B of these examinations will be standardized in the near future, however, under a project that is being undertaken jointly

by the War Department and the Cooperative Test Service of the American Council on Education.

It is to be expected that various schools will desire to determine supplementary data on Form B of these tests for their own information and use. Such a procedure would be highly desirable, and it is to be hoped that representative colleges and universities will secure copies of Form B of these examinations for this purpose.

It is to be emphasized that the official responsibilities of this Subcommittee are limited to the above-mentioned subject examinations. The examinations in mathematics at the high school level were under the supervision of consultants nominated by officers of the National Council of Teachers of Mathematics. Outside the province of the Subcommittee are also certain end-of-course tests in mathematics at the college level prepared for the U. S. A. F. I.; these examinations are non-accreditation examinations for correspondence courses and class instruction on mathematics courses offered by the Institute.

The procedure of examination construction followed by the Subcommittee was as follows:

(a) One member of the committee assumed the responsibility of constructing the examination in a given course;

(b) subsequent to the construction of an examination, two other members served as official critics, going over thoroughly all questions and responses, and offering criticisms and suggestions;

(c) after the member responsible for the construction had made revisions in the light of the criticisms of the official critics, the examination was submitted to the group as a whole for further criticism, and, finally, for approval.

Following are descriptions of the individual examinations. The examinations strive to test the examinee's appreciation and understanding of the fundamental concepts, as well as his skill in techniques.

ALGEBRA—COLLEGE LEVEL

Sec. I. Fundamental Operations, Radicals, Exponents, and Linear Equations in One Unknown. 13 items.
Sec. II. Systems of Linear Equations in One and Two Unknowns; Quadratic Equations and Related Topics. 20 items.
Sec. III. Binomial Theorem, Progressions, Variations, and Complex Numbers. 14 items.
Sec. IV. Inequalities and the Theory of Equations. 14 items.
Sec. V. Determinants, Permutations, Combinations, and Probability. 9 items.

PLANE TRIGONOMETRY

Sec. I. Right Triangles and Applications. 10 items.
Sec. II. Logarithms and the use of Tables. 10 items.

Sec. III. Oblique Triangles and Applications. 9 items.
Sec. IV. General Angle, Variation of Functions, Graphs. 14 items.
Sec. V. Identities and Equations. 17 items.

ANALYTIC GEOMETRY

Sec. I. Points and Lines. 15 items.
Sec. II. General Loci and the Conic Sections. 28 items.
Sec. III. Transformation of Coordinates; Parametric Equations. 9 items.
Sec. IV. Polar Coordinates. 11 items.
Sec. V. Space Geometry. 12 items.

[If examinee is qualified to take only Secs. I–IV on Plane Analytic Geometry, it is recommended that examination time be reduced to 2 hours, 25 min.]

CALCULUS I. DIFFERENTIAL CALCULUS (NO INTEGRATION INCLUDED)

Sec. I. Functions; Definition and Interpretation of Derivatives. 11 items.
Sec. II. Formal Differentiation. 15 items.
Sec. III. Applications of Differentiation. 24 items.
Sec. IV. Infinite Series; Partial Derivatives. 16 items.

CALCULUS II. INTEGRAL CALCULUS (FOLLOWING INITIAL COURSE CONTAINING NO INTEGRATION)

Sec. I. Indefinite Integrals. 25 items.
Sec. II. Definite Integrals. 25 items.
Sec. III. Double and Triple Integrals. 10 items.

Section IV of the above described examination in Calculus I is issued as a supplement, thus permitting interchange with an alternate Sec. IV, containing 16 problems on integration. With this interchange, there is provided the following examination:

CALCULUS I. DIFFERENTIAL CALCULUS (INCLUDING SIMPLE INTEGRATION)

Sec. I. Functions; Definition and Interpretation of Derivatives. 11 items.
Sec. II. Formal Differentiation. 15 items.
Sec. III. Applications of Differentiation. 24 items.
Sec. IV. Integration. 16 items.

The provision of this alternate form of the examination in the first course in Calculus seemed desirable in view of the varied manner in which Calculus is taught throughout the country. An accompanying examination for the one in Calculus I, including simple integration, might be obtained by deleting sixteen problems from Secs. I and II of the above described examination in Calculus II, and adjoining the supplement on Infinite Series and Partial Derivatives. If a man has taken enough work under the auspices of the U.S.A.F.I. to present himself for examination in the two courses in Calculus described above, however, it is advised that he take Calculus I, as first described, and Calculus II, although the institution to which he goes may present the topics in calculus in a different order.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-eighth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado State College of Education, Greeley, Colorado, on April 14 and 15, 1944. There were three sessions, the final session being a joint meeting with the Mathematics Section of the Eastern Division of the Colorado Education Association. Professor A. E. Mallory, Chairman of the Section, presided at each of the sessions.

The attendance was thirty-two, including the following twelve members of the Association: C. F. Barr, A. G. Clark, J. C. Fitterer, Leota C. Hayward, A. J. Kempner, Claribel Kendall, W. J. LeVeque, A. J. Lewis, A. E. Mallory, Greta Neubauer, A. W. Recht, E. C. Varnum.

At the business meeting the following officers were elected for the coming year: Chairman, Jack Britton, University of Colorado; Vice-Chairman, C. F. Barr, University of Wyoming.

The following papers were presented:

1. *Mathematics in the A. S. T. P.*, by Professor A. G. Clark, Colorado State College of Agriculture and Mechanic Arts.

2. *Vibrating membranes*, by August Newlander, University of Denver, introduced by the Secretary.

It was pointed out that the vibrations of a circular membrane are in many respects similar to the vibrations of a string. The displacement of a particular point of the membrane can be obtained by solving a partial differential equation of the second order. The solution of the differential equation can be expressed by means of an infinite series involving sines, cosines, and Bessel functions. The determination of certain constants dependent upon the boundary conditions involves the use of Fourier series and the Fourier-Bessel expansion of a function. After all constants have been determined, the result is an expression giving the displacement of any point of the membrane in terms of its position and the time after releasing the membrane from rest.

3. *Reduction of inverse tangents to integral arguments*, by Professor E. C. Varnum, University of Wyoming.

By a study of the operation $(a-b)/(1-ab)$ the speaker developed formulas by which inverse tangents of rational arguments may be reduced to those having integral arguments, the latter having been well tabulated in recent projects.

4. *A new definition of the Gamma function*, by Mrs. Margaret Matchett, University of Denver, introduced by the Secretary.

The speaker remarked that the Gamma function is uniquely defined by its

functional equation and the condition of logarithmic convexity. It was also stated that this definition yields an explicit expression for the Gamma function as an infinite product.

5. *Suggested changes in the content of high school mathematics*, by Ruth Hoffman, Denver Public Schools, introduced by the Secretary.

6. *Trends in grade placement of arithmetic fundamentals*, by L. B. Garner, Cameron School, Greeley, Colorado, introduced by Professor Mallory.

7. *I know better than I teach, now*, by Professor A. W. Recht, University of Denver.

In this address it was suggested that every teacher constantly fails to reach standards of teaching which he knows to be better. Twelve points for good teaching were submitted with the suggestion that they be used for periodic check-ups. One of these points, that of keeping the student informed of his standing day by day, was explained in detail. A method was shown by which the daily running averages of students in a whole class could be written down in two or three minutes from one setting of a slide rule.

8. *Early computation with Hindoo-Arabic numbers*, by Professor A. E. Mallory, Colorado State College of Education.

A. J. LEWIS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Eighth Summer Meeting, Montreal, Canada, June 23–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May, 1945
METROPOLITAN NEW YORK, Brooklyn, April 21, 1945
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in November 1944 to Professor R. H. Cameron for his paper, "Some introductory exercises in the manipulation of Fourier transforms," published in the *National Mathematics Magazine,* vol. 15 (1941), pp. 331-356.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

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